Rational Points on Curves over Finite Fields: Bounds, Geometry, and Maximality

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new structures with **Hilbert**
and to finish by sheaves and schemes with **Grothendieck** and **Deligne**.

Historical

I. A little historical overview on the development of algebraic geometry

(after Jean Dieudonné)

Prehistory

400 before J.-C. - 1630 after J.-C.

Prehistory

For the Greeks, Algebra is essentially "geometrical".

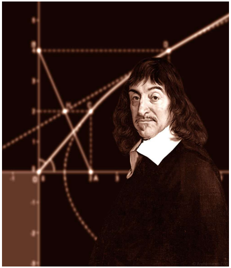
Method of resolution of **algebraic problems** which consist to obtain a solution by the **intersection** of auxiliary **curves**.

Exploration

1630 - 1795

Exploration : René Descartes (31 mars 1596 - 11 février 1650)

Invention of the **cartesians coordinates**, due independently to Descartes and Fermat.



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Exploration : René Descartes (31 mars 1596 - 11 février 1650)

The invention of cartesians coordinates gives a systematic process of **translation** of any **geometric** relation between points on the plane or in the space into a relation between the **coordinates** of this points !

This gives rise to the **birth** of algebraic geometry and differential geometry !

Exploration : René Descartes (31 mars 1596 - 11 février 1650)

The first invariant notion which appears is that of *degree* : Descartes knows that the **degree** of an algebraic plane curve is **invariant** under axes changes.

Exploration

The general conception of **parametric representations** of a curve is the main idea behind Newton on the infinitesimal Calculus.

The study of the **tangents** to a curve $F(x, y) = 0$ leads to the consideration of **singular points** (Newton, Leibniz, and then Cramer).

Number of **intersection points** on two plane algebraic curves : Maclaurin, Euler and Bézout.

Third period (1795 - 1850)

The golden age of projective geometry

(1795 - 1850)

Third period (1795 - 1850)

Monge (1795) and then Poncelet systematically introduce some **points at infinity** and some **imaginary points**.

One works in the complex **projective plane** $\mathbb{P}^2(\mathbb{C})$ or the 3-dimensional complex **projective space** $\mathbb{P}^3(\mathbb{C})$.

The **intersection of two circles** in the plane is no longer an exception to the Bézout theorem for the conics : the number of points on the intersection is now equal to 4 in $\mathbb{P}^2(\mathbb{C})$ if we consider the imaginary points and the points at infinity.

Third period (1795 - 1850) - suite

The use of the **homogeneous coordinates** by **Möbius**, **Plücker** and **Cayley**.

But some people (**Chasles**, **Steiner**) don't want to use them (they focus on cross-ratio, theory of foci...).

It is the research of the **pure geometry** which implies some "principles" more or less fuzzy (denounced by **Cauchy**).

The mirage of the "purity" has **delayed** the acknowledgement of the capital role of **Linear and Multilinear** Algebra in Geometry.

Third period (1795 - 1850) - suite

Introduction of the n-**dimensional** space in 1845 by **Grassmann** and **Cayley**.

But the general study of the varieties contained in any projective spaces $\mathbb{P}^n(\mathbb{C})$ **didn't** begin before 1885 with **Segre**.

Fourth period (1850 - 1866)

Riemann and the birational geometry (1850 - 1866)

Fourth period (1850 - 1866)

Riemann introduced the theory of **abelian integrales** :

$$u = \int_a^x \frac{R(t)dt}{\sqrt{P(t)}}$$

where $P(t)$ is a polynomial of degree 3 or 4 and $R(t)$ is a rational function.

They give for example the **length** of an arc of ellipse or of lemniscaste.

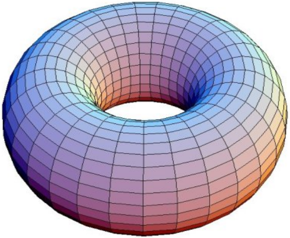
Fourth period (1850 - 1866)

This theory leads into the notion of **Riemann surface** : topological space covered by opens U_i which are homeomorphic to opens V_i of $\mathbb{C}$, the homeomorphisms ϕ_i satisfying a chart gluing condition (when $U_i \cap U_j \neq \emptyset$ then $\phi_i \circ \phi_j^{-1}$ is an holomorphic function on $\phi_j(U_i \cap U_j)$).

An example of compact Riemann surface is the **Riemann sphere** $\mathbb{P}^1(\mathbb{C})$ obtained from $\mathbb{C}$ by adding a point at infinity.

Fourth period (1850 - 1866)

A Riemann surface of genus 1 :



Riemann associates to any compact Riemann surface S a **field**, called after **Dedekind**, the rational functions field of S.

Fourth period (1850 - 1866)

Two Riemann surfaces are considered to be **equivalent** if there exists a bijective and biholomorphic correspondence between them.

The associated curves are **birationally equivalent** (the functions fields are isomorphic).

It's the advent of the **birational geometry** which will going to dominate all the algebraic geometry during 80 years (the projective invariants give place to the **birational invariants** (as the genus)).

Fifth period (1866 - 1920)

Development and chaos (1866 - 1920)

Fifth period (1866 - 1920)

Riemann disappears at 40 years old in 1866.
Its work inspired **several schools** in Algebraic Geometry which have tendance to diverge :

- the **geometrical** school with the linear series (Brill and Max Noether in Germany, Smith and Cayley in England, Halphen in France, Zeuthen in Danemark and the first generation of italians geometers Cremona, Segre and Bertini).

- the **transcendantal** theory of algebraic varieties (Clebsch, Picard, Lefschetz).

- the **algebraic** school.

Fifth period (1866 - 1920)

Leopold Kronecker

He dreams on a big construction which would include the **number theory** and the **algebraic geometry**.

Fifth period (1866 - 1920)

Kronecker defines in 1882 an **algebraic variety** as the set of zeros in $\mathbb{C}^n$ of a family of polynomials $P_\alpha \in \mathbb{C}[T_1, \ldots, T_n]$.

He precises the notion of **irreducible** variety (if the ideal of the ring $\mathbb{C}[T_1, \ldots, T_n]$ generated by the P_α is prime) ;

and that of **dimension** (transcendance degree over $\mathbb{C}$ of the function field).

Fifth period (1866 - 1920)

Richard Dedekind and Heinrich Weber

The article of Dedekind-Weber in 1882 focus on algebraic curves and give **purely algebraic proofs** of all the theorems of Riemann.

Fifth period (1866 - 1920)

During the years 1870-1880, Dedekind creates the theory of **divisibility** in number fields.

The **ring of integers** A of a number field K (set of elements of K which are integral over $\mathbb{Z}$) is a Dedekind domain (integrally closed Noetherian domain in which every nonzero prime ideals is maximal).

A fractional ideal of K is a A-module $\mathfrak{a}$ contained in K and such that there exists a nonzero c in A such that $c\mathfrak{a} \in A$.

The **fractional ideals** of K write on the form

$$\mathfrak{P}_1^{\alpha_1} \dots \mathfrak{P}_r^{\alpha_r}$$

where the $\mathfrak{P}_i$ are prime ideals of A and the $\alpha_i \in \mathbb{Z}$.

Fifth period (1866 - 1920)

Dedekind and Weber generalize the notion of fractional ideal by introducing the notion of **divisor** on the function field K of the curve :

$$D = \sum n_P P$$

where the points P of the curve correspond to the discrete valuations of K and the n_P are almost all zero rational integers.

The thing that **generalize** a fractional ideal is the set $L(D)$ of $f \in K$ such that

$$v_P(f) \geq -n_P$$

for all point P on the curve.

Fifth period (1866 - 1920)

The sets $L(D)$ are finite dimensional vector spaces whose dimension $\ell(D)$ is given by the **Riemann-Roch** theorem :

$$\ell(D) = deg(D) + 1 - g + \ell(W - D)$$

where g is the genus and W the canonical divisor of the curve.

Sixth period (1920 - 1950)

New structures in algebraic geometry

(1920 - 1950)

Sixth period (1920 - 1950)

The development of **Abstract Algebra** (structures of groups, rings, fields, modules...) around 1900 drive to extend the concepts of algebraic geometry on **any** (commutative) fields (non necessarily algebraically closed and non necessarily of characteristic zero).

Sixth period (1920 - 1950)

We need a **general theory** of fields (definition of transcendance degree, existence for any field of an algebraic closure...) : **Steinitz** in 1910.

Transcendantal separable extensions (indispensable to the development of algebraic geometry over a field of characteristic $p > 0$) : **MacLane** in 1939.

Sixth period (1920 - 1950)

The theory of rings in the polynomials rings is thorough by **Emmy Noether**

following the results of Hilbert (in 1890-1893) as :

the **finite basis theorem** (if A is a Noetherian commutative ring then $A[X_1, \ldots, X_n]$ is Noetherian)

and the **theorem of zeros** (it's the Nullstellensatz : if K is an algebraically closed field and I is an ideal of $K[X_1, \ldots, X_n]$ then $\mathcal{I}(Z(I)) = \sqrt{I}$).

Sixth period (1920 - 1950)

Field of definition : since 1940, we always precise that a variety is defined over a field k.

Chow and **Van der Waerden** consider the points on the variety over all the extensions K of k.

Krull studies in detail the **local** rings.

Zariski studies **singularities** (normal varieties (local rings integrally closed), resolution of singularities) and defines a **topology** on any k-variety (the closed sets are the k-subvarieties).

Sixth period (1920 - 1950)

In his thesis in 1921, **Emil Artin** has observed that the algebraic congruences

$$F(x, y) \equiv 0 \pmod{p}$$

can be interpreted as algebraic equations over the **prime field** $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

Moreover, the analogy between the finite extensions of $\mathbb{C}(X)$ and the number fields is much deeper when we replace $\mathbb{C}$ by $\mathbb{F}_q$ (since the residue fields of the valuations of a finite extension of $\mathbb{F}_q(X)$ are finite fields as for the valuations of a number field).

Sixth period (1920 - 1950)

The Riemann zeta function : the definition

It is defined for all $s \in \mathbb{C}$ with real part > 1 by the **Dirichlet serie** :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and has an analytic continuation to the whole complex plane minus 1.

It writes as an **Eulerian product** :

$$\zeta(s) = \prod_{p \ prime} \frac{1}{1 - p^{-s}}.$$

Sixth period (1920 - 1950)

The Riemann zeta function : the functional equation

The function ζ satisfies the following functional equation for all complex s different to 0 and 1 :

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s)$$

where

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt.$$

Sixth period (1920 - 1950)

The Riemann Hypothesis

Besides the trivial zeros (in $s = -2n$), one conjectures that the **zeros** of the function ζ have all a **real part** $1/2$.

Sixth period (1920 - 1950)

The zeta functions

Artin introduces the analogue of the Riemann-Dedekind zeta function of number fields :

$$\sum_{\mathfrak{a}} \mid N(\mathfrak{a}) \mid^{-s} = \prod_{\mathfrak{P}} (1 - \mid N(\mathfrak{P} \mid^{-s})^{-1}$$

where $\mathfrak{a}$ runs through the set of integral ideals of K and $\mathfrak{P}$ the set of prime ideals, and N denotes the norm.

Sixth period (1920 - 1950)

The zeta function of curves and varieties over finite fields

The zeta function of a projective algebraic curve (or variety) X defined over $\mathbb{F}_q$ is defined by :

$$Z_X(T) = \exp\Big(\sum_{i=1}^{\infty} \sharp X(\mathbb{F}_{q^n}) \frac{T^n}{n}\Big).$$

Sixth period (1920 - 1950)

Example : The zeta function of the projective line over $\mathbb{F}_q$

Since $\sharp\mathbb{P}^1(\mathbb{F}_{q^n}) = q^n + 1$ for any $n \geq 1$ then :

$$Z_X(T) = \exp\Big(\sum_{i=1}^{\infty}(q^n+1)\frac{T^n}{n}\Big)$$

$$= \exp\Big(\sum_{i=1}^{\infty}\frac{(qT)^n}{n}\Big) \times \exp\Big(\sum_{i=1}^{\infty}\frac{T^n}{n}\Big)$$

$$= \frac{1}{1-qT} \times \frac{1}{1-T}.$$

Sixth period (1920 - 1950)

Rationality of the zeta function of a "nice" curve (F.K. Schmidt, 1931)

$$Z_X(T) = \frac{L_X(T)}{(1-T)(1-qT)}$$

where $L_X(T)$ is a polynomial of $\mathbb{Z}[T]$ of degree $2g$ (where g is the genus of X) of the form :

$$L_X(T) = \prod_{i=1}^{g}(1-\omega_i T)(1-\bar{\omega}_i T).$$

Sixth period (1920 - 1950)

Functional equation

F.K. Schmidt find that the functional equation

$$Z_X(1/qT) = q^{1-g}T^{2-2g}Z_X(T)$$

comes from the Riemann-Roch theorem for X.

Sixth period (1920 - 1950)

Riemann Hypothesis for curves

It enables to prove that for $1 \leq i \leq g$:

$$\mid \omega_i \mid = \sqrt{q}.$$

Proved by **Helmut Hasse** in 1933 for $g = 1$, and in 1940 by **André Weil** for all g.

Sixth period (1920 - 1950)

Riemann Hypothesis for curves versus the conjecture for the Riemann zeta function

We define the Dedekind-zeta function of the curve X by :

$$\zeta_X(s) := \prod_{P \ closed \ point \ on \ X} (1 - N(P)^{-s})^{-1}$$

where $N(P)$=number of elements of the residue field of P.

On can show that

$$\zeta_X(s) = Z_X(q^{-s}).$$

Hence

$$|\omega_i| = \sqrt{q} \Longleftrightarrow Re(s) = \frac{1}{2}.$$

Sixth period (1920 - 1950)

Weil bounds

The Riemann Hypothesis implies immediately that :

$$\mid \sharp X(\mathbb{F}_{q^n}) - (q^n + 1) \mid \leq 2gq^{n/2}.$$

Sixth period (1920 - 1950)

Weil's conjectures

The Riemann hypothesis of higher dimensional varieties.

Seventh period (1950 -)

Sheaves and schemes

(1950 -)

Seventh period (Sheaves and schemes)

Considerables advances after the introduction of the notions of **sheaves**, cohomology with coefficients in a sheaf and **spectral sequences** invented by **Leray**.

The **Serre varieties** : in 1954 **Jean-Pierre Serre** use the notion of "ringed space" : topological space X on which is given a sheaf of rings $\mathcal{O}_X$, called **structural sheaf**
(for all open set $U \subset X$, $H^0(U, \mathcal{O}_X)$ has a ring structure such that the restriction homomorphisms $H^0(U, \mathcal{O}_X) \longrightarrow H^0(V, \mathcal{O}_X)$ for $V \subset U$ are ring homomorphisms).

Seventh period (Sheaves and schemes)

Alexandre Grothendieck, around 1957, begins a huge program to generalize algebraic geometry.

He starts from the **category of all commutative rings** instead of the subcategory of finitely generated reduced algebras over an algebraically closed field.

He replaces the affine varieties by the **spectrum** $Spec(A)$ of a ring A (set of all prime ideals of A) together with a **Zariski topology** (the closed sets are the $V(\mathfrak{a})$ i.e. the set of prime ideals containing an ideal $\mathfrak{a}$).

Seventh period (Sheaves and schemes)

The goal : define a “good” cohomology theory.

Georges de Rham (do not confuse with **Alexander Rahm** !) introduced a cohomology based on the existence of differential forms with prescribed properties.

The hypercohomology of the de Rham complex of sheaves has been introduced by Grothendieck and is related to crystalline cohomology (developped by **Pierre Berthelot**).

Seventh period (Sheaves and schemes)

Around 1963, using the cohomology with values in a sheaf for the **étale topology**, Grothendieck defines, for any prime ℓ different from the characteristic of k, ℓ-**adic étale cohomological spaces**

$$H^i(\overline{X}, \mathbb{Q}_\ell)$$

over the field $\mathbb{Q}_\ell$ of ℓ-adic numbers.

He shows with **M. Artin** that this cohomology has all the good wanted properties (Künneth formula, Poincaré duality, Lefschetz trace formula...).

Seventh period (Sheaves and schemes)

Let X be an algebraic variety over $k = \mathbb{F}_q$ and

$$\overline{X} = X \times_k \overline{k}$$

the variety after **base change extension** to the algebraic closure $\overline{k}$ of k.

Let ℓ be a prime number distinct from the characteristic of k.

The **groups** $H^i(\overline{X}, \mathbb{Q}_\ell)$ are $\mathbb{Q}_\ell$-vector spaces which are zero for $i < 0$ and $i > 2\dim X$.

Seventh period (Sheaves and schemes)

The **Frobenius morphism** $F : \overline{X} \longmapsto \overline{X}$ sends the point P of coordinates (x_i), $x_i \in \overline{k}$, on the point $F(P)$ of coordinates (x_i^q).

It is clear that P is a fixed-point of F iff $P \in X(k)$. More generally : P is a fixed-point of F^n iff $P \in X(k_n)$. By the **Lefschetz fixed-point formula**, we obtain :

$$\sharp X(k_n) = \sum_{i=0}^{2 \dim X} (-1)^i \, Tr(F^n \mid H^i(\overline{X}, \mathbb{Q}_\ell)).$$

Seventh period (Sheaves and schemes)

Substituing in the zeta function, we get :

$$Z_X(T) = \prod_{i=0}^{2\dim X} [\exp(\sum_{n=1}^{\infty} Tr(F^n \mid H^i(\overline{X}, \mathbb{Q}_\ell))\frac{T^n}{n})]^{(-1)^i}.$$

The **Grothendieck-Lefschetz trace-formula** gives an **interpretation** of the **zeta function** of X defined over a finite field k in terms of the (reciprocal polynomials) of the **characteristic polynomial** of the Frobenius endomorphism F induced on the étale ℓ-adic cohomology groups with compact support of $\overline{X}$.

Seventh period (Sheaves and schemes)

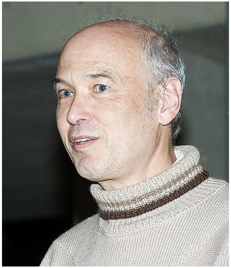
For example, for a **curve** X, we get :

$$Z_X(T) = \frac{P_{k,H^1(X)}(T)}{P_{k,H^0(X)}(T)\ P_{k,H^2(X)}(T)}$$

where

$$P_{k,H^i(X)}(T) = \det(1 - FT \mid H_c^i(\overline{X}, \mathbb{Q}_\ell)).$$

Seventh period (Sheaves and schemes)



In 1973, **Pierre Deligne proved the Weil conjectures** (in particular the Riemann hypothesis for varieties of any dimension) using all the theory of schemes of Grothendieck.

He was awarded the Fields Medal in 1978 :

"Deligne gave solution of the three Weil conjectures concerning generalizations of the Riemann hypothesis to finite fields. His work did much to unify algebraic geometry and algebraic number theory."

II. Maximal curves over finite fields

Neron-Severi group and Hodge-index theorem

The **Neron-Severi group** of the surface $X \times X$ can be quotiented by numerical equivalence and thus tensorised to obtain a real vector space

$$\mathrm{Num}(X \times X)_{\mathbb{R}}$$

equipped by the intersection pairing.
As a consequence of the **Hodge-index theorem** the intersection pairing is negative definite on the vector space orthogonal to the plane generated by the classes of the horizontal and vertical fibres. We denote by $\wp$ the orthogonal projection onto this subspace. For an integer k we thus consider γ_k the push-down by $\wp_*$ of the class of the graph of the k-th iterated **Frobenius** that we normalize by $\sqrt{q}^k$.

Weil bound

The non-negativity of the determinant of the **Gram matrix** $\mathrm{Gram}(\gamma_0, \gamma_1)$ expresses exactly the **Weil inequality** :

$$| \sharp X(\mathbb{F}_q) - (q+1) | \leq 2g\sqrt{q}. \tag{1}$$

Ihara bound

Next, the non-negativity of the Gram matrix Gram$(\gamma_0, \gamma_1, \gamma_2)$ together with the arithmetic constraint $\sharp X(\mathbb{F}_{q^2}) \geq \sharp X(\mathbb{F}_q)$ yields to the **Ihara bound** (1981) :

$$\sharp X(\mathbb{F}_q) - (q+1) \ \leq \frac{\sqrt{(8q+1)g^2 + (4q^2-4q)g} - g}{2}. \tag{2}$$

Hallouin-Perret bound

Meanwhile, **Hallouin and Perret** in 2019 have also noticed that the non-negativity of the determinant of the Gram matrix $\mathrm{Gram}(\gamma_0, \gamma_1, \gamma_2)$ leads to the following inequality :

$$\sharp X(\mathbb{F}_{q^2}) - (q^2 + 1) \leq 2gq - \frac{1}{g}\big(\sharp X(\mathbb{F}_q) - (q + 1)\big)^2. \qquad (3)$$

Weil-maximal curves

A Weil-maximal curve is a curve X which attains the Weil bound i.e. such that

$$\sharp X(\mathbb{F}_q) = (q+1) + 2g\sqrt{q}$$

(so q has to be a square !).

Ihara has proved that if X is Weil-Maximal then $g \leq \frac{\sqrt{q}(\sqrt{q}-1)}{2}$.

The Hermitian curve

The **Hermitian curve** over $\mathbb{F}_q$ (where q is a square), whose equation is

$$x^{\sqrt{q}+1} + y^{\sqrt{q}+1} + z^{\sqrt{q}+1} = 0$$

is Weil-maximal with maximal genus $g = \frac{\sqrt{q}(\sqrt{q}-1)}{2}$.

But we have more : ‘

Theorem (Rück-Stichtenoth (1994))

We suppose that q is a square and we consider a curve X defined over $\mathbb{F}_q$. Suppose that X has genus $g = \frac{\sqrt{q}(\sqrt{q}-1)}{2}$. Then X is Weil-maximal **if and only if** *X is $\mathbb{F}_q$-isomorphic to the Hermitian curve.*

The study of Hallouin-Perret-maximal curves

New results

(joint work with Fabien Herbaut and Julien Monaldi)

to appear in the International Journal of Number Theory (2026)

Hallouin-Perret-maximal curves : the definition

Definition

A curve X defined over $\mathbb{F}_q$ of genus $g > 0$ is said to be a Hallouin-Perret-maximal curve (or HP-maximal curve for short) if

$$\sharp X(\mathbb{F}_{q^2}) - (q^2 + 1) = 2gq - \frac{1}{g}\left(\sharp X(\mathbb{F}_q) - (q + 1)\right)^2. \qquad (4)$$

Hallouin-Perret-maximal curves : elliptic curves

It is straighforward that any **elliptic curve** E over $\mathbb{F}_q$ is a HP-maximal curve since

$$N_1 := \sharp E(\mathbb{F}_q) = q + 1 - (\omega + \overline{\omega})$$

and

$$N_2 := \sharp E(\mathbb{F}_{q^2}) = q^2 + 1 - (\omega^2 + \overline{\omega}^2)$$

and thus

$$(N_1 - (q+1))^2 = (\omega + \overline{\omega})^2 = q^2 + 1 + 2q - N_2.$$

Hallouin-Perret-maximal curves : caracterization by the zeta function

Proposition (A-Herbaut-Monaldi (2026))

Let X be a curve of genus g defined over $\mathbb{F}_q$. We denote by $\alpha_1, \dots, \alpha_g$ the real parts of its Frobenius eigenvalues. The curve X is a Hallouin-Perret-maximal curve **if and only if** $\alpha_1 = \cdots = \alpha_g$, *that is if and only if its zeta function is of the form*

$$Z_X(T) = \frac{(1 - 2\alpha T + qT^2)^g}{(1 - T)(1 - qT)}$$

where α is the common value of all the α_j. In this case, 2α is an integer and we have $2\alpha = \frac{q+1-\sharp X(\mathbb{F}_q)}{g}$.

Hallouin-Perret-maximal curves : Weil-max and Weil-min curves

As a consequence of the previous proposition, a **Weil-maximal** curve (respectively a **Weil-minimal** curve) over $\mathbb{F}_q$ is a **HP-maximal** curve.

Indeed, if X is a Weil-maximal curve over $\mathbb{F}_q$ of genus g then $\sharp X(\mathbb{F}_q) = q + 1 + 2g\sqrt{q}$ and q must be a square. If we denote by $\omega_1, \overline{\omega}_1, \ldots, \omega_g, \overline{\omega}_g$ its Frobenius eigenvalues, then we can write $\sharp X(\mathbb{F}_q) = q + 1 - \sum_{j=1}^{g}(\omega_j + \overline{\omega}_j)$ and we thus obtain that $\sum_{j=1}^{g}(\omega_j + \overline{\omega}_j) = -2g\sqrt{q}$. But the Riemann Hypothesis says that $|\omega_j| = \sqrt{q}$ for any $j = 1, \ldots, g$ which implies that all the ω_j are equal (to $-\sqrt{q}$) and then X is a HP-maximal curve.

In a same way, if X is Weil-minimal then all the ω_j are equal (to $\sqrt{q}$) and then X is also a HP-maximal curve.

Hallouin-Perret-maximal curves : other examples

A HP-maximal curve of genus $g \geq 2$ is not necessarily Weil-maximal nor Weil-minimal :

the curve X of genus 2 defined over $\mathbb{F}_3$ by the equation

$$y^2 = (-1 - x - x^3)(1 - x + x^3)$$

verifies $N_1 := \sharp X(\mathbb{F}_3) = 2$ and $N_2 := \sharp X(\mathbb{F}_9) = 20$ so X is a **HP-maximal** curve but is **neither** Weil-maximal **nor** Weil-minimal.

Hallouin-Perret-maximal curves : a geometric condition

Proposition (A-Herbaut-Monaldi (2026))

Let X be a curve of genus g defined over $\mathbb{F}_q$. If X is a Hallouin-Perret-maximal curve then the Jacobian of X is $\mathbb{F}_q$-isogenous to a **power** *of a $\mathbb{F}_q$-***simple abelian variety***.*

Abelian varieties

Let A be an abelian variety of dimension g defined over $\mathbb{F}_q$.
The characteristic polynomial of the Frobenius endomorphism on the $\mathbb{Q}_\ell$-vector space $T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is a monic polynomial of degree $2g$ with integer coefficients whose roots $\omega_1, \overline{\omega}_1, \ldots, \omega_g, \overline{\omega}_g$ have all modulus $\sqrt{q}$ (the Riemann Hypothesis for abelian varieties proved by Weil).
If we set $\tau_A := -\sum_{j=1}^{g}(\omega_j + \overline{\omega}_j) = -2\sum_{j=1}^{g} \alpha_j$ for the opposite of the trace of the Frobenius on A (where the α_j's are the real parts of the complex numbers ω_j's), then **Aubry, Haloui and Lachaud** proved in 2013 the following bound on the number of rational points on A :

$$\sharp A(\mathbb{F}_q) \leq (q + 1 + \tau_A/g)^g \tag{5}$$

with equality if and only if the α_j's are equal.

Hallouin-Perret-maximal curves : a characterization by the number of points of the Jacobian

Proposition (A-Herbaut-Monaldi (2026))

Let X be a curve of genus g defined over $\mathbb{F}_q$. Then X is a Hallouin-Perret-maximal curve **if and only if** *the number of rational points of the Jacobian* $\mathrm{Jac}(X)$ *of X attains the upper bound (5), namely*

$$\sharp\,\mathrm{Jac}(X)(\mathbb{F}_q) = (q + 1 + \tau_{\mathrm{Jac}(X)}/g)^g. \tag{6}$$

Hallouin-Perret-maximal curves : bounded genus

Proposition (A-Herbaut-Monaldi (2026))

Let X be a Hallouin-Perret-maximal curve defined over $\mathbb{F}_q$ of genus g. Then

$$g \leq 23q^2 \log q.$$

Moreover, when we write $L_X(T) = (1 + 2\alpha T + qT^2)^g$ then

1. *If $2\alpha > 0$ then $g < (\sqrt{q} + 1)^4 (\frac{q^2+1}{2q^2})$.*
2. *If $2\alpha = 2\sqrt{q}$ then $g \leq \frac{q-\sqrt{q}}{2}$.*
3. *If $2\alpha = -2\sqrt{q}$ then $g \leq \frac{(\sqrt{q}+1)^2}{2\sqrt{q}}$.*

Hallouin-Perret-maximal curves : in coverings

Proposition (A-Herbaut-Monaldi (2026))

Let $Y \longrightarrow X$ be a non-constant morphism of curves over $\mathbb{F}_q$. If Y is a Hallouin-Perret-maximal curve then X is also a Hallouin-Perret-maximal curve.

Diophantine-stables curves

Recall that we have obviously $\sharp X(\mathbb{F}_{q^2}) \geq \sharp X(\mathbb{F}_q)$.

Following **Mazur-Rubin** (2018) a curve satisfying

$$\sharp X(\mathbb{F}_{q^2}) = \sharp X(\mathbb{F}_q)$$

is called a **Diophantine-stable** curve (with respect to the extension $\mathbb{F}_{q^2}/\mathbb{F}_q$).

Ihara-maximal curves : the defintion

Definition

A curve X defined over $\mathbb{F}_q$ of genus $g > 0$ is said to be a **Ihara-maximal** curve if

$$\sharp X(\mathbb{F}_q) - (q+1) = \frac{\sqrt{(8q+1)g^2 + (4q^2-4q)g} - g}{2}. \qquad (7)$$

Ihara-maximal curves : the characterization

Proposition (A-Herbaut-Monaldi (2026))

Let X be a curve of genus $g \geq 1$ defined over $\mathbb{F}_q$. The following assertions are **equivalent**.

1. *X is a Ihara-maximal curve.*
2. *X is both a Hallouin-Perret-maximal curve and a Diophantine-stable curve with respect to the extension $\mathbb{F}_{q^2}/\mathbb{F}_q$.*
3. $Z_X(T) = \frac{(1-2\alpha T+qT^2)^g}{(1-T)(1-qT)}$ *where* $\alpha = \frac{1}{4} - \frac{\sqrt{(8q+1)g^2+(4q^2-4q)g}}{4g}$.

Ihara-maximal curves : bounded genus

Hallouin and Perret proved in 2019 that if a curve is Ihara-maximal then its genus is

$$g \leq \frac{\sqrt{q}(q-1)}{\sqrt{2}}.$$

Ihara-maximal curves of maximal genus

One can state an analogous result of those of **Rüch-Stichtenoth** (1994) about Weil-maximal curves of maximal genus.
This is a kind of reformulation of a result of **Fuhrmann-Torres** (1998).

Theorem (A-Herbaut-Monaldi (2026))

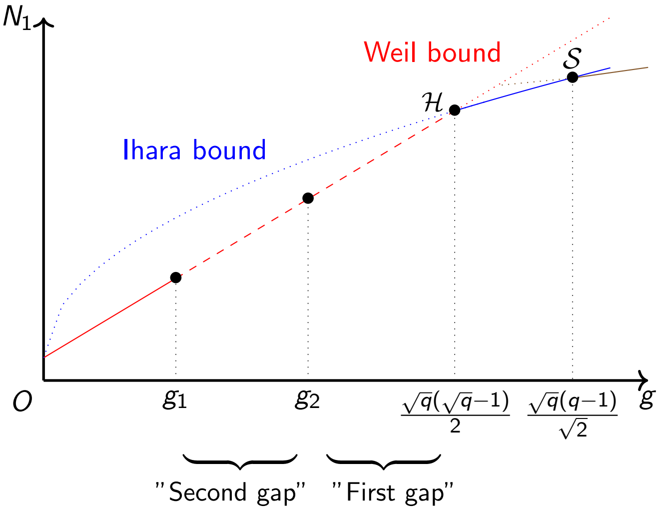
We consider $t \geq 1$ and $q = 2^{2t+1}$. Let X be a curve defined over $\mathbb{F}_q$. Suppose that X has genus $g = \frac{\sqrt{q}(q-1)}{\sqrt{2}}$. Then X is Ihara-maximal **if and only if** *X is $\mathbb{F}_q$-isomorphic to the Suzuki curve S which is the non-singular model of the curve of equation $y^q - y = x^{q_0}(x^q - x)$ where $q_0 = 2^t$.*

Ihara-maximal curves of maximal genus : the proof

Démonstration.

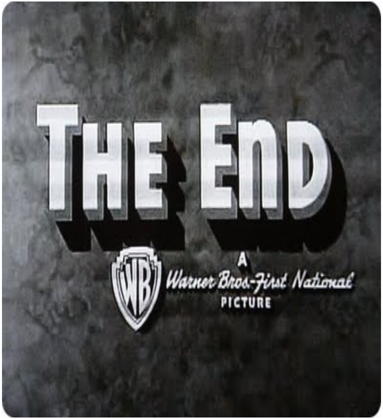
A curve X defined over $\mathbb{F}_q$ (with $q = 2^{2t+1}$) of genus $g = \frac{\sqrt{q}(q-1)}{\sqrt{2}}$ is Ihara-maximal if and only if $\#X(\mathbb{F}_q) = q^2 + 1$.
But Fuhrmann and Torres proved that if $q = 2^{2t+1}$ and $q_0 = 2^t$ then any curve of genus $g = q_0(q-1)$ and such that $\#X(\mathbb{F}_q) = q^2 + 1$ is isomorphic to the Suzuki curve.
Since $q_0(q-1) = \frac{\sqrt{q}(q-1)}{\sqrt{2}}$, the result follows. □

Ihara-maximal curves of maximal genus : the picture



The end

Thank you for your attention



Annexe : One proof

HP-maximality in a covering

Proposition (A-Herbaut-Monaldi (2026))

Let $Y \longrightarrow X$ be a non-constant morphism of curves over $\mathbb{F}_q$. If Y is a Hallouin-Perret-maximal curve then X is also a Hallouin-Perret-maximal curve.

Proof - 1/2

The numerator $L_X(T)$ of the zeta function of X is the reciprocal polynomial of the characteristic polynomial of the Frobenius endomorphism on $T_\ell(J_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. The map

$$f^* : J_X \longrightarrow J_Y$$

induced by f on the Jacobians has finite kernel and sends the ℓ^n-torsion points of J_X on those of J_Y. Then, tensorising by $\mathbb{Q}_\ell$, we get an injective morphism of $\mathbb{Q}_\ell$-vector spaces

$$T_\ell(J_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \xrightarrow{f^* \otimes 1} T_\ell(J_Y) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Proof - 2/2

The Frobenius morphism on $T_\ell(J_Y) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ leaves fixed the subspace $T_\ell(J_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Hence the characteristic polynomial of the former divides the characteristic polynomial of the latter in $\mathbb{Q}_\ell[T]$, hence in $\mathbb{Z}[T]$ since both $L_X, L_Y \in \mathbb{Z}[T]$ have constant term equals to 1. Thus, we have that $L_X(T)$ divides $L_Y(T)$.

So, if Y is HP-maximal then all the real parts of its Frobenius eigenvalues are equal. Since $L_X(T)$ divides $L_Y(T)$, we get also that all the real parts of the Frobenius eigenvalues of X are equal and thus X is also HP-maximal.