THE JACOBI BOUND CONJECTURE AND THE DIMENSION CONJECTURE

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BASIC QUESTION:

- Given system of non-linear ODEs in several dependent variables how many constants of integration do you need?
- How many "arbitrary" functions?

$x_1(t), \ldots, x_{10}(t)$ 10 functions of complex variable

$$\begin{cases} u_1 = 0, u_2 = 0 \\ u_3 = 0, u_4 = 0 \\ u_5 = 0, u_6 = 0 \end{cases}$$ 6 ODEs

degree = 5, order 2

$$u_i = F_i(x_1, x_1', x_1'', x_2, x_2', x_2'', \ldots, x_{10}, x_{10}', x_{10}'')$$

polynomial

$$\begin{cases} u_1=0,\ u_2=0 \\ u_3=0,\ u_4=0 \\ u_5=0,\ u_6=0 \end{cases}$$

degree = 5, $x_1(t), \dots, x_{10}(t)$

order 2

Question: How many arbitrary function will the solution depend on?

Question: How many constants of integration will the solution depend on?

2-algebraic variety

$$\Sigma : \begin{cases} u_1 = 0,\ u_2 = 0 \\ u_3 = 0,\ u_4 = 0 \\ u_5 = 0,\ u_6 = 0 \end{cases}$$

<u>Question</u>: How many arbitrary function will the solution depend on?

<u>Question</u>: How many constants of integration will the solution depend on?

Question: How many arbitrary function will the solution depend on?

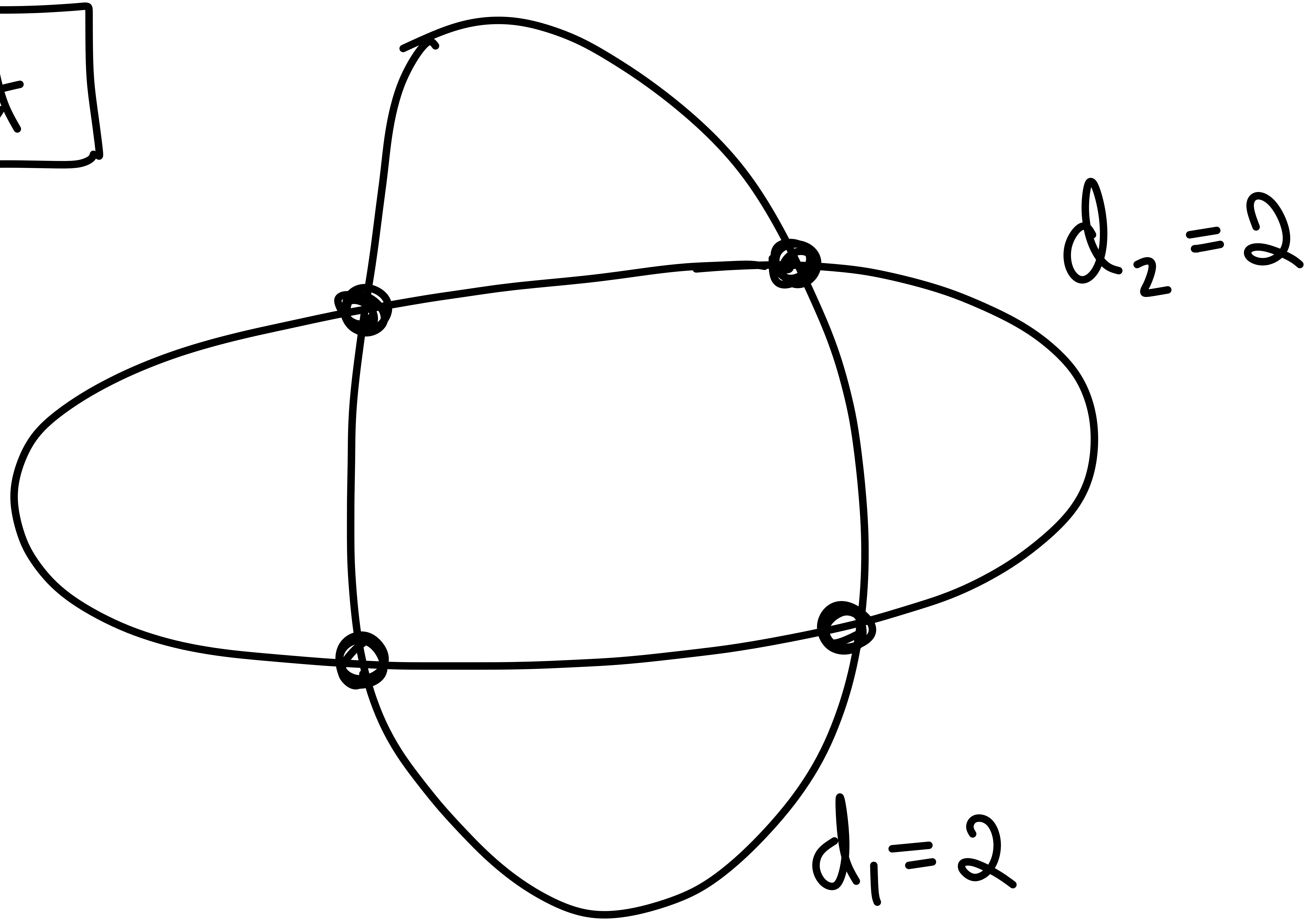
$\rightsquigarrow \dim^{\partial}(\Sigma)$ = differential dimension
= # arbitrary functions

Question: How many constants of integration will the solution depend on?

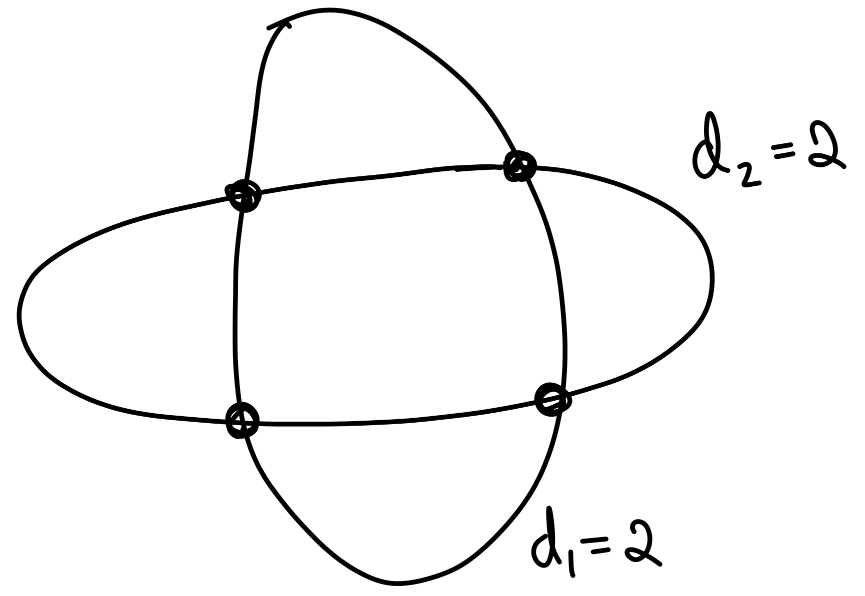
$\rightsquigarrow \dim_{\mathbb{K}}(\Sigma)$ = classical dimension
= # constants of integration

JACOBI BOUND CONJECTURE

Bézout



Bézout



Let $f_1, f_2, \dots, f_n \in K[x_1, x_2, \dots, x_n]$.

Let $X = V(f_1, f_2, \dots, f_n) \subseteq \mathbb{A}^n$.

$$\dim(X) = 0 \implies \text{length}_K(X) \le d_1 d_2 \cdots d_n$$

where $d_i = \deg(f_i)$

Jacobi Bound Conjecture

Let $u_1, u_2, \ldots, u_n \in K\{x_1, x_2, \ldots, x_n\}$.

Let $\Sigma = V([u_1, u_2, \ldots, u_n]) \subseteq \mathbb{A}^n_\infty = \operatorname{Spec} K\{x_1, x_2, \ldots, x_n\}$

Let Σ_1 irred component of Σ.

$$\dim^\partial(\Sigma_1) = 0 \Rightarrow \dim(\Sigma_1) \leq J(u_1, u_2, \ldots, u_n)$$

Example: $K\{x\} = K[x, x', x'', \dots]$ — ring of ∂ poly

$= \mathcal{O}(\mathbb{A}^1_\infty)$ — arc space of $\mathbb{A}^1$!

Look at $u = x'' \in K\{x\}$,

$[u] = \langle x'', x''', x'''', \dots \rangle$ — ∂-ideal generated by u.

then

$$\frac{K\{x\}}{[u]} = \frac{K[x, x', x'', \dots]}{\langle x'', x''', x'''', \dots \rangle} \cong K[x, x'].$$

Solve ODE:

$$u = x'' = 0$$

order 2

$$\operatorname{trdeg}_K(K[x, x']) = 2$$

transcendence degree 2

Solution: $$\frac{d^2x}{dt^2} = 0$$

$$\Rightarrow x(t) = at + b$$

2 constants of integration

Example: $K\{x,y\} = K[x, x', x'', \ldots, y, y', y'', \ldots]$

∂-polynomial ring

$$\begin{cases} u_1 = x'' + 2y^2 + 1 = 0 \\ u_2 = y''' + x' = 0 \end{cases}$$

order matrix

$$A = \begin{pmatrix} \mathrm{ord}_x^{\partial}(u_1) & \mathrm{ord}_y^{\partial}(u_1) \\ \mathrm{ord}_x^{\partial}(u_2) & \mathrm{ord}_y^{\partial}(u_2) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

$$\frac{K\{x,y\}}{[u_1,u_2]} \cong \frac{K\{x\}[y,y',y'']}{[x''+2y^2+1]} \cong K[x,x',y,y',y'']$$

$\langle u_1, \partial(u_1), \dots, u_2, \partial(u_2), \dots \rangle$ ← smallest ∂-ideal containing u_1 & u_2.

$$\begin{cases} u_1 = x''+2y^2+1 = 0 \\ u_2 = y'''+x' = 0 \end{cases}$$

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

$$\frac{K\{x,y\}}{[u_1,u_2]} \cong K[x,x',y,y',y'']$$

$$\begin{cases} \partial(y'') = -x' \\ \partial(x') = -2y^2-1 \end{cases}$$

has structure of ∂-ring

$$\begin{cases} u_1 = x''+2y^2+1 = 0 \\ u_2 = y'''+x' = 0 \end{cases}$$

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

$$\frac{K\{x,y\}}{[u_1,u_2]} \cong K[x,x',y,y',y'']$$ ← transcendence degree 5

$\mathrm{tdet}(A)$ tropical determinant.

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

$$\mathrm{tdet}(A) = J = \max\{2+3, 0+1\} = 5$$

"Jacobi Bound"

JACOBI BOUND CONJECTURE

Jacobi Bound Conjecture

Let $u_1, u_2, \ldots, u_n \in K\{x_1, x_2, \ldots, x_n\}$.

Let $\Sigma = V([u_1, u_2, \ldots, u_n]) \subseteq \mathbb{A}^n_\infty = \operatorname{Spec} K\{x_1, x_2, \ldots, x_n\}$

Let Σ_1 irred component of Σ.

$$\dim(\Sigma_1) < \infty \Longrightarrow \dim(\Sigma_1) \leq J(u_1, u_2, \ldots, u_n)$$

Jacobi Bound Conjecture

Let $u_1, u_2, \dots, u_n \in K\{x_1, x_2, \dots, x_n\}$.

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Let Σ_1 irred component of Σ.

$$\dim(\Sigma_1) < \infty \implies \dim(\Sigma_1) \leq J(u_1, u_2, \dots, u_n)$$

$$J(u_1, u_2, \dots, u_n) = \text{Jacobi Bound} = \operatorname{tdet}(A)$$

$$A = \begin{pmatrix} \operatorname{ord}^{\partial}_{x_1}(u_1) & \operatorname{ord}^{\partial}_{x_2}(u_1) & \cdots & \operatorname{ord}^{\partial}_{x_n}(u_1) \\ \operatorname{ord}^{\partial}_{x_1}(u_2) & \operatorname{ord}^{\partial}_{x_2}(u_2) & \cdots & \operatorname{ord}^{\partial}_{x_n}(u_2) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{ord}^{\partial}_{x_1}(u_n) & \operatorname{ord}^{\partial}_{x_2}(u_n) & \cdots & \operatorname{ord}^{\partial}_{x_n}(u_n) \end{pmatrix}$$

$$\operatorname{tdet}(A) = \max_{\rho \in S_n} \sum_{i=1}^{n} \operatorname{ord}^{\partial}_{x_i}(u_{\rho(i)})$$

Jacobi Bound Conjecture

Let $u_1, u_2, \ldots, u_n \in K\{x_1, x_2, \ldots, x_n\}$.

Let $\Sigma = V([u_1, u_2, \ldots, u_n]) \subseteq \mathbb{A}^n_\infty = \operatorname{Spec} K\{x_1, x_2, \ldots, x_n\}$

Let Σ_1 irred component of Σ. $\dim(\Sigma_1) < \infty$

$$\dim(\Sigma_1) \leq \max_{\rho \in S_n} \sum_{i=1}^{n} \operatorname{ord}^{\partial}_{x_i}(u_{\rho(i)})$$

maximorum transversalium systemate praeditum evadat. Solutio exemplo illustratur.

Variabilem independentem vocemus t, eius functiones sive variabiles pro dependentibus habitas $x_1, x_2, \ldots, x_m$; inter quas variabiles propositae sint m aequationes differentiales

$$u_1 = 0, \quad u_2 = 0, \quad \ldots, \quad u_m = 0.$$

Sit $a_{i,x}$ ordo altissimi, quod in aequatione $u_i = 0$ obvenit, differentialis variabilis x_x, dico:

1) ordinem systematis aequationum differentialium propositarum sive numerum Constantium arbitrariarum, quem earum integratio completa poscit, aequari *maximo* inter omnes valores, quos aggregatum

$$a_{i_1,1} + a_{i_2,2} + \cdots + a_{i_m,m}$$

induat, si pro indicibus $i_1, i_2, \ldots, i_m$, quibuscunque modis fieri potest, sumantur m diversi ex indicibus $1, 2, \ldots, m$.

... systematis ordinem designabo per O; aequabitur O summae

DE AEQUATIONUM DIFFERENTIALIUM SYSTEMATE NON NORMALI AD FORMAM NORMALEM REVOCANDO*).

(Ex Ill. C. G. J. Jacobi manuscriptis posthumis in medium protulit A. Clebsch.)

In commentatione mea „Theoria novi Multiplicatoris etc.“ **) Multiplicatorem determinavi aequationum differentialium *isoperimetricarum*, i. e. ad problemata illa isoperimetrica pertinentium, in quibus variatio dati integralis, variabilem unam independentem, ceteras dependentes continentis, ad nihilum redi-
ritur. Quam determinationem multo maioribus difficultatibus obnoxiam esse

HISTORY:

- 1865 Jacobi
- 1935 Ritt
- 1969, 1970, 1972 Lando
- 1976 Tomasovic
- 2004 Hrushovski
- 2007 Ollivier-Sadik
- 2008, 2009 KMP
- 2014 Li-Li

Example: $\begin{cases} u_1 = xy' \\ u_2 = x''(y^2+1) \end{cases}$, $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, $J = 3$

$\Sigma = V([u_1, u_2])$ has ≥ 4 components

$$[xy', x''(y^2+1)] = [x, x''] \cap [x, y^2+1] \cap [y', x''] \cap [y, y^2+1]$$

$$= P_1 \cap P_2 \cap P_3 \cap P_4$$

$$[xy', x''(y^2+1)] = [x, x''] \cap [x, y^2+1] \cap [y', x''] \cap [y, y^2+1]$$
$$= P_1 \cap P_2 \cap P_3 \cap P_4$$

, $J = 3$

Look at Σ_1:

$$\Sigma_1 = \operatorname{Spec} \frac{K\{x,y\}}{P_1} = \operatorname{Spec} K\{y\}$$

infinite dim'l

Look at Σ_3:

$$\Sigma_3 = \operatorname{Spec} \frac{K\{x,y\}}{P_3} = \operatorname{Spec} K[x, x', y]$$

3 dim'l.

Strong vs Weak JBC

$\mathrm{ord}_y^{\theta}(u_1) = 0$

$$\begin{cases} x' = 0 \\ x^{(99)} + (y')^2 = 0 \end{cases}$$

$\mathrm{ord}_y^{\theta}(u_1) = -\infty$

weak

strong

$$A = \begin{bmatrix} 1 & 0 \\ 99 & 1 \end{bmatrix}$$

weak $J = 99$

$$A = \begin{bmatrix} 1 & -\infty \\ 99 & 1 \end{bmatrix}$$

strong $J = 2$

weak

strong

} ⇝

TWO FLAVORS OF JBC:

Weak JBC

Strong JBC

Easy Theorem

Strong JBC $\implies$ Weak JBC

Easy Theorem

Strong JBC $\implies$ Weak JBC

Easy Theorem:

Strong JBC $\implies$ DC

Easy Theorem:

strong JBC $\Rightarrow$ DC

Dimension Conjecture (DC)

If $u_1, \dots, u_m \in K[x_1, \dots, x_n]$ with $m < n$ then every component of $\Sigma = V([u_1, \dots, u_m])$ is infinite dim'l.

[Easy Theorem:
strong JBC $\Rightarrow$ DC

- Suppose Strong JBC
- Let $u_1, u_2 \in K\{x, y, z\}$.
- Let $\Sigma = V([u_1, u_2, 0])$ and let Σ_1 be an irreducible component.
- If $\dim(\Sigma_1) < \infty \Rightarrow \dim(\Sigma_1) \leq J$.

$$\dim(\Sigma_1) \le J, = +\det(A) = -\infty.$$

$$A = \begin{pmatrix} * & * & * \\ * & * & * \\ -\infty & -\infty & -\infty \end{pmatrix}$$

strong

$$\Sigma = V([u_1, u_2, 0])$$

$$\dim(\Sigma_1) \leq J, = \operatorname{tdet}(A) = -\infty,$$

JBC: $\forall\, \Sigma_1 \in \mathrm{Irr}(\Sigma)$

$$\dim(\Sigma_1) < \infty \Rightarrow \dim(\Sigma_1) \leq -\infty.$$

no component with $\dim(\Sigma_1) < \infty$ can have $\dim(\Sigma_1) \leq -\infty$!

Conclusion: There are no finite dim'l components of $\Sigma = V([u_1, u_2, 0]) \subseteq \mathbb{A}^3_\infty$.

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General Statement:
If $u_1, \ldots, u_m \in K\{x_1, \ldots, x_n\}$ with $m < n$ then every component of $\Sigma = V([u_1, \ldots, u_m])$ is infinite dim'l.

General Statement:

If $u_1, \ldots, u_m \in K[x_1, \ldots, x_n]$ with $m < n$ then every component of $\Sigma = V([u_1, \ldots, u_m])$ is infinite dim'l.

Dimension Conjecture (DC)

If $u_1, \ldots, u_m \in K[x_1, \ldots, x_n]$ with $m < n$ then every component of $\Sigma = V([u_1, \ldots, u_m])$ is infinite dim'l.

Easy Theorem:

strong JBC $\Longrightarrow$ DC

New Results

(joint w/ David Zureick-Brown)

THEOREM:

JBC holds for generically reduced components.

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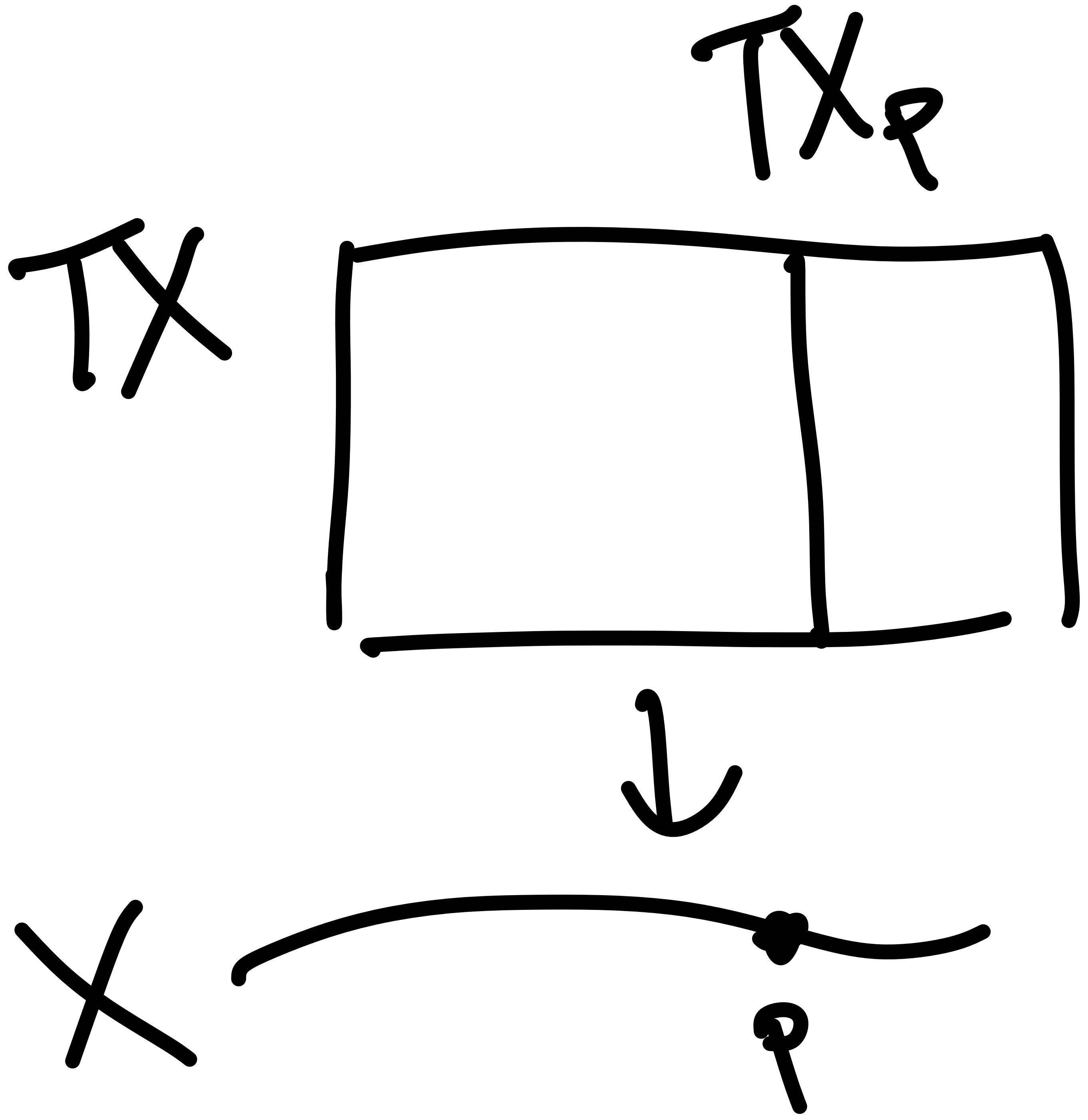
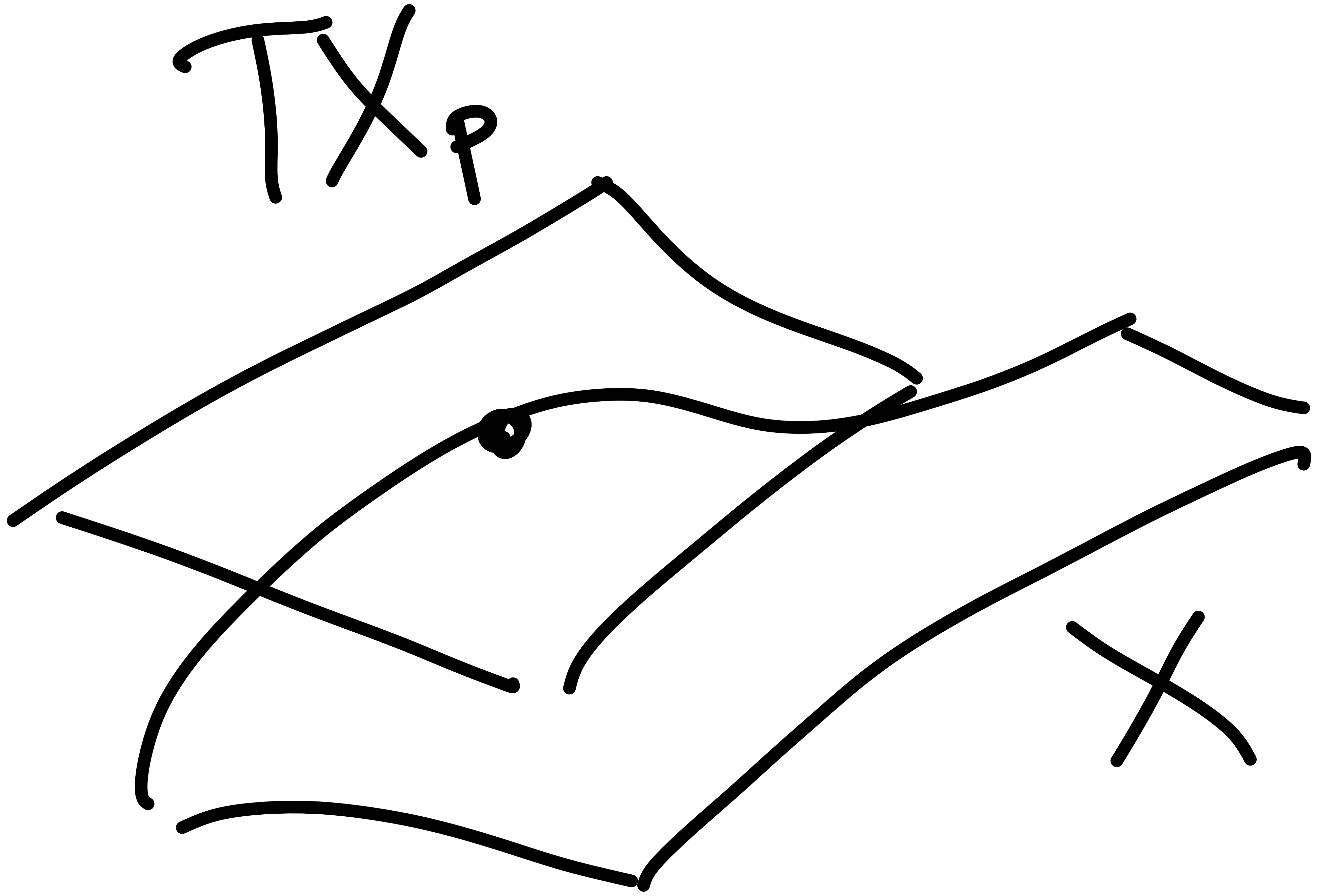
Tangent Bundles

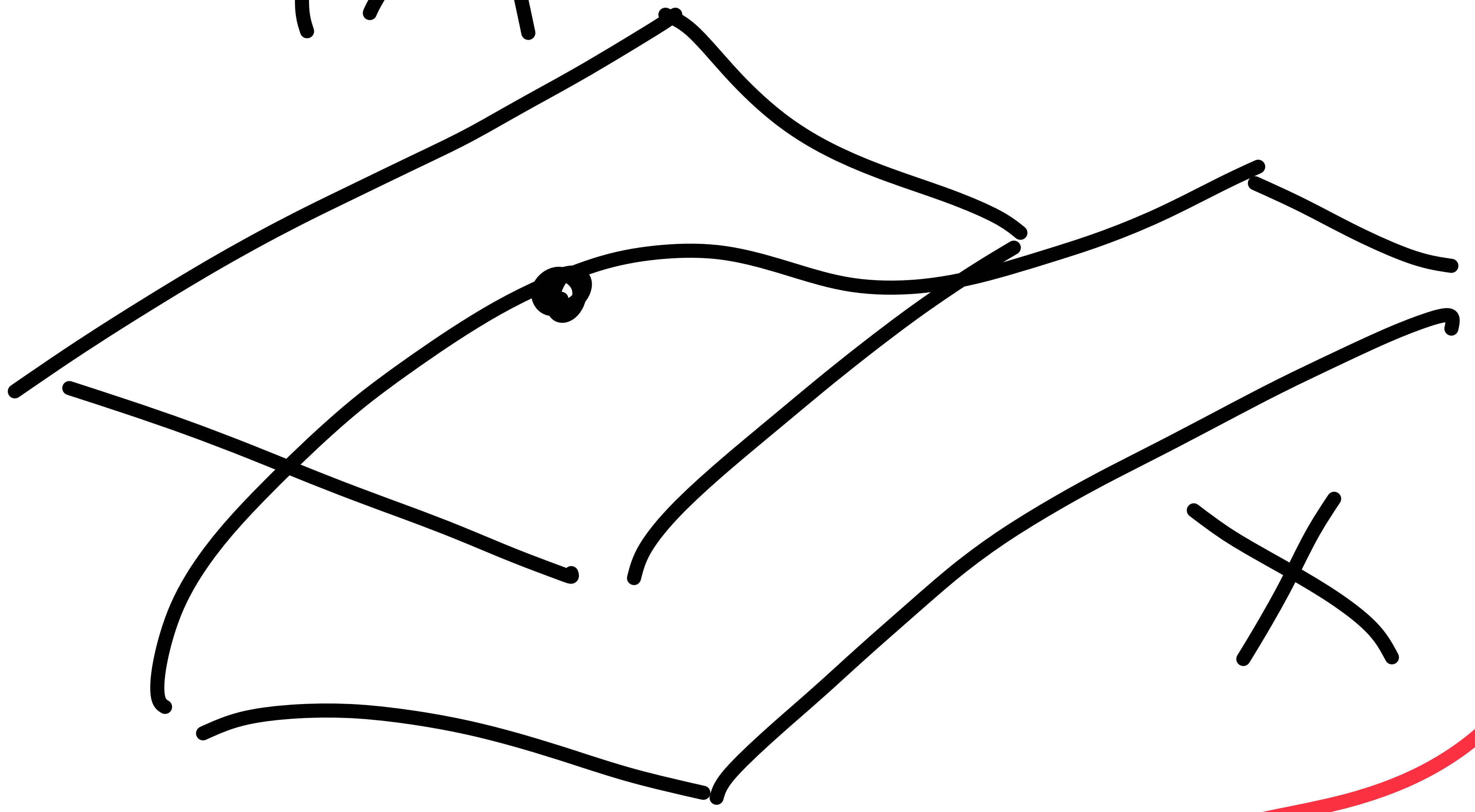




Tangent Bundles

$$\begin{cases} X: \ x^2 + y^2 - 1 = 0 \\ p = (x_0, y_0) \end{cases}$$

TX_p



$$\begin{cases} x = x_0 + \varepsilon x_1, \\ y = y_0 + \varepsilon y_1. \end{cases} \qquad \varepsilon^2 = 0$$

$$(x_0 + \varepsilon x_1)^2 + (y_0 + \varepsilon y_1)^2 - 1 = 0$$

Tangent Bundles

$$\begin{cases} X: \; x^2+y^2-1=0 \\ p=(x_0,y_0) \end{cases}$$

$$(x_0+\varepsilon x_1)^2+(y_0+\varepsilon y_1)^2-1=0$$

$$\Rightarrow (x_0^2+y_0^2-1)+\varepsilon(2x_0x_1+2y_0y_1)=0$$

$$TX: \begin{cases} x_0^2+y_0^2-1=0 \\ 2x_0x_1+2y_0y_1=0 \end{cases}$$

Tangent Bundles

$$\begin{cases} X: \; x^2 + y^2 - 1 = 0 \\ p = (x_0, y_0) \end{cases} \Rightarrow TX: \begin{cases} x_0^2 + y_0^2 - 1 = 0 \\ 2x_0x_1 + 2y_0y_1 = \end{cases}$$

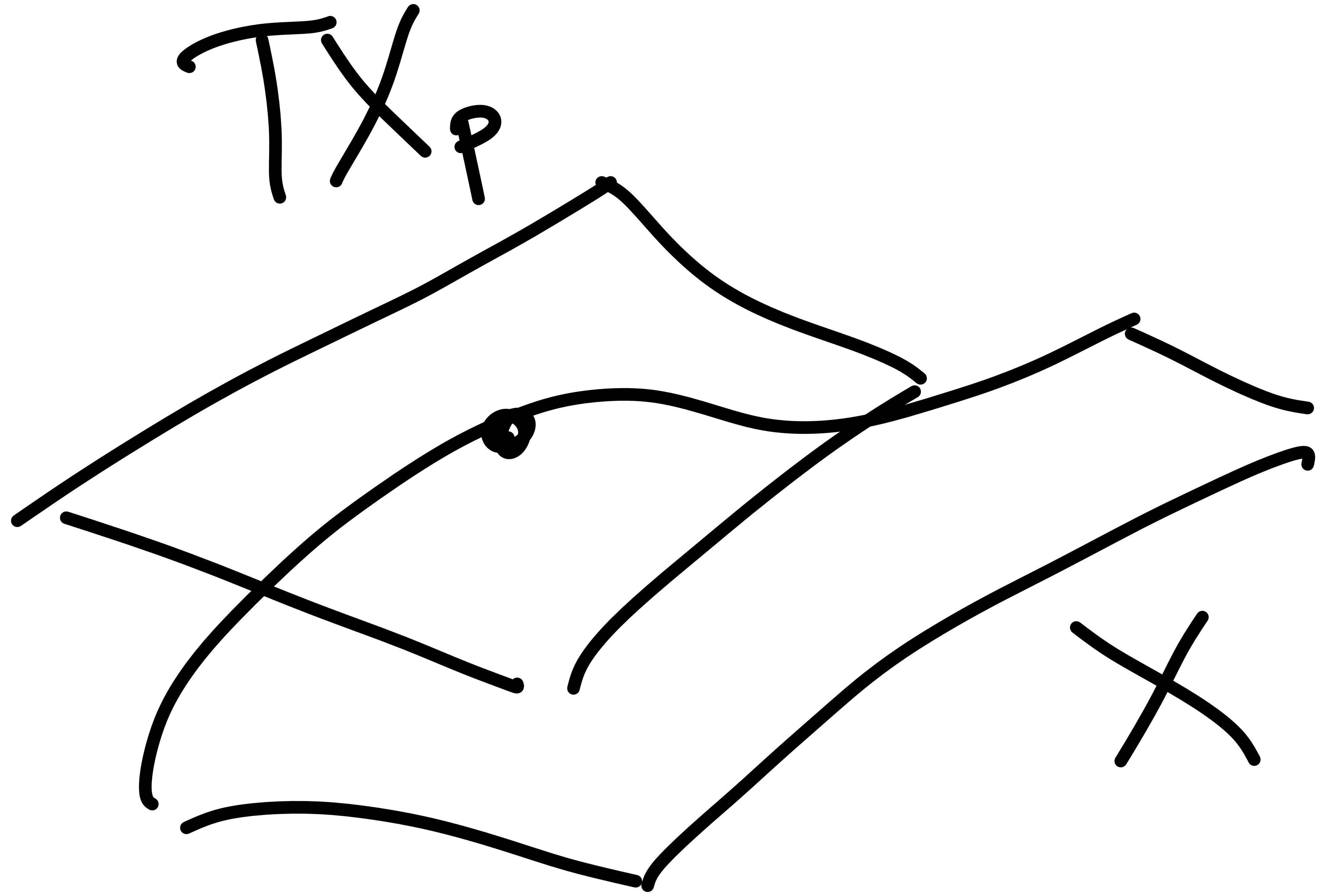
Tangent Bundles

$$\begin{cases} X: \quad x^2 + y^2 - 1 = 0 \\ p = (x_0, y_0) \end{cases} \Rightarrow TX: \begin{cases} x_0^2 + y_0^2 - 1 = 0 \\ 2x_0x_1 + 2y_0y_1 = \end{cases}$$

Tangent Bundles

$$TX : \begin{cases} x_0^2 + y_0^2 - 1 = 0 \\ 2x_0x_1 + 2y_0y_1 = 0 \end{cases}$$

$$\begin{cases} X : \ x^2 + y^2 - 1 = 0 \\ p = (x_0, y_0) \end{cases} \Rightarrow$$



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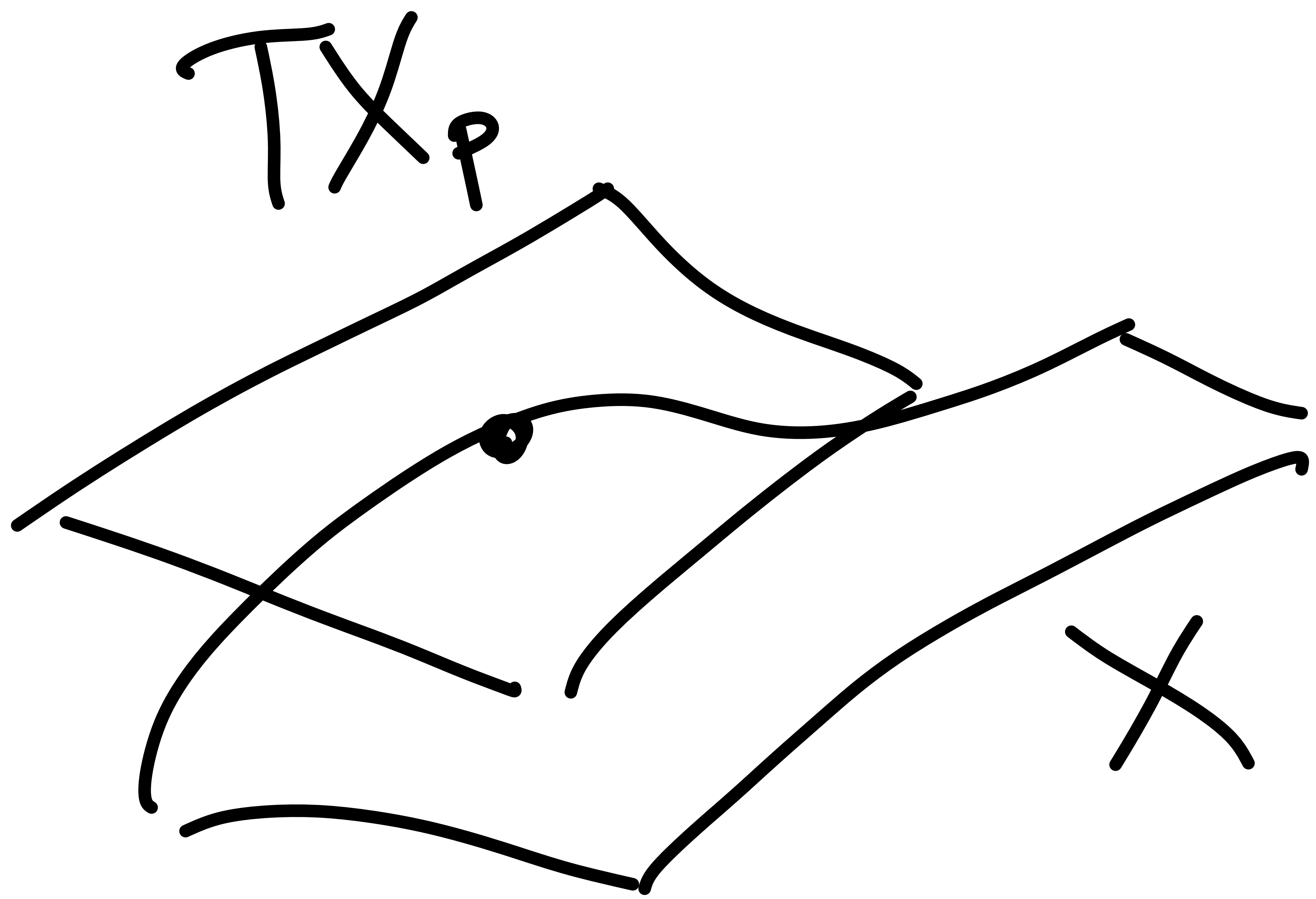

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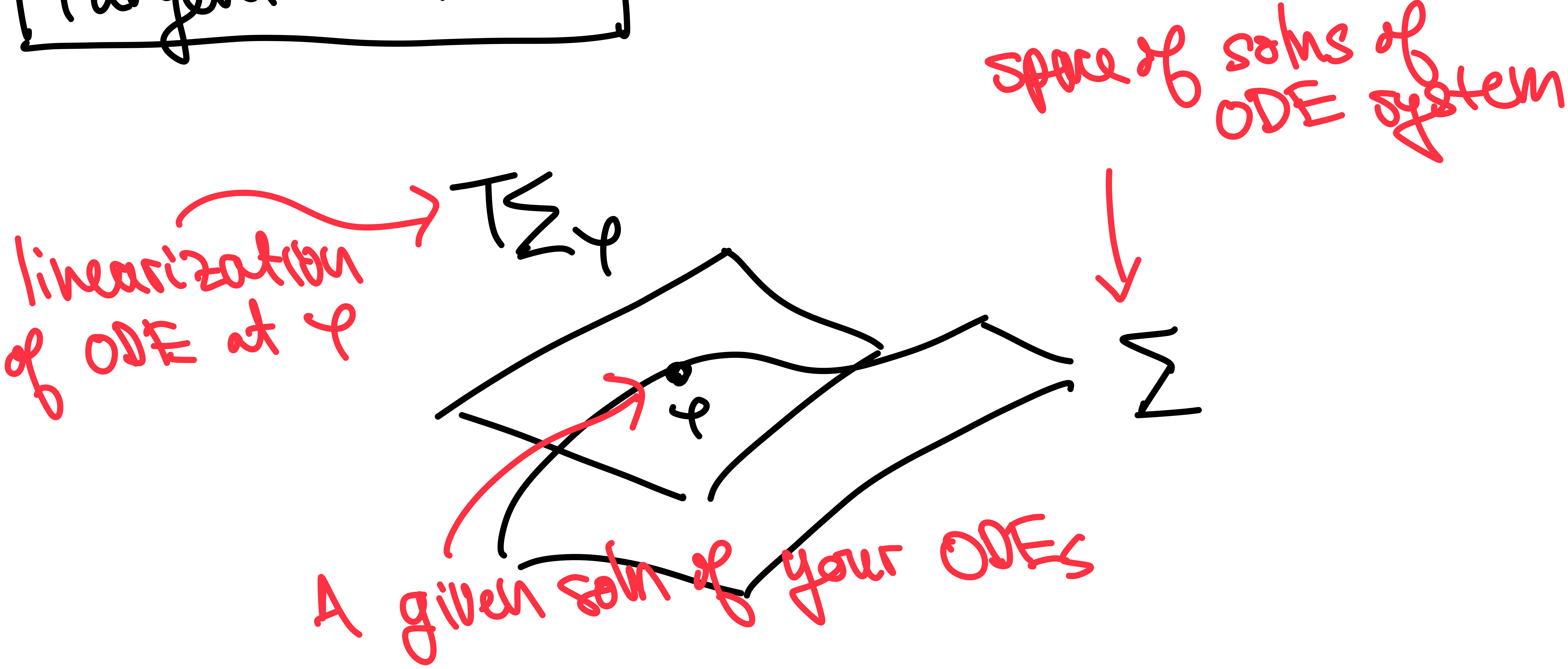

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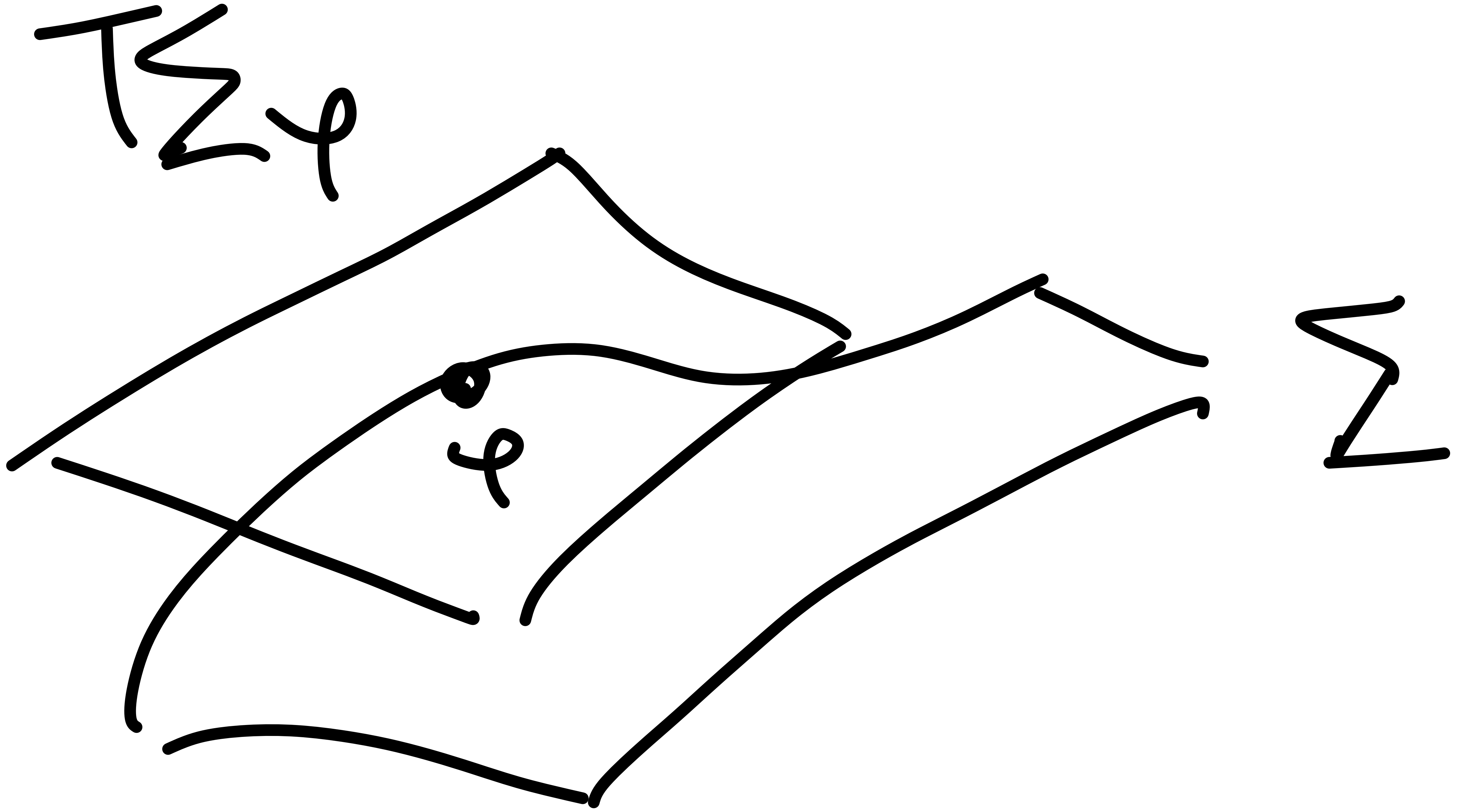

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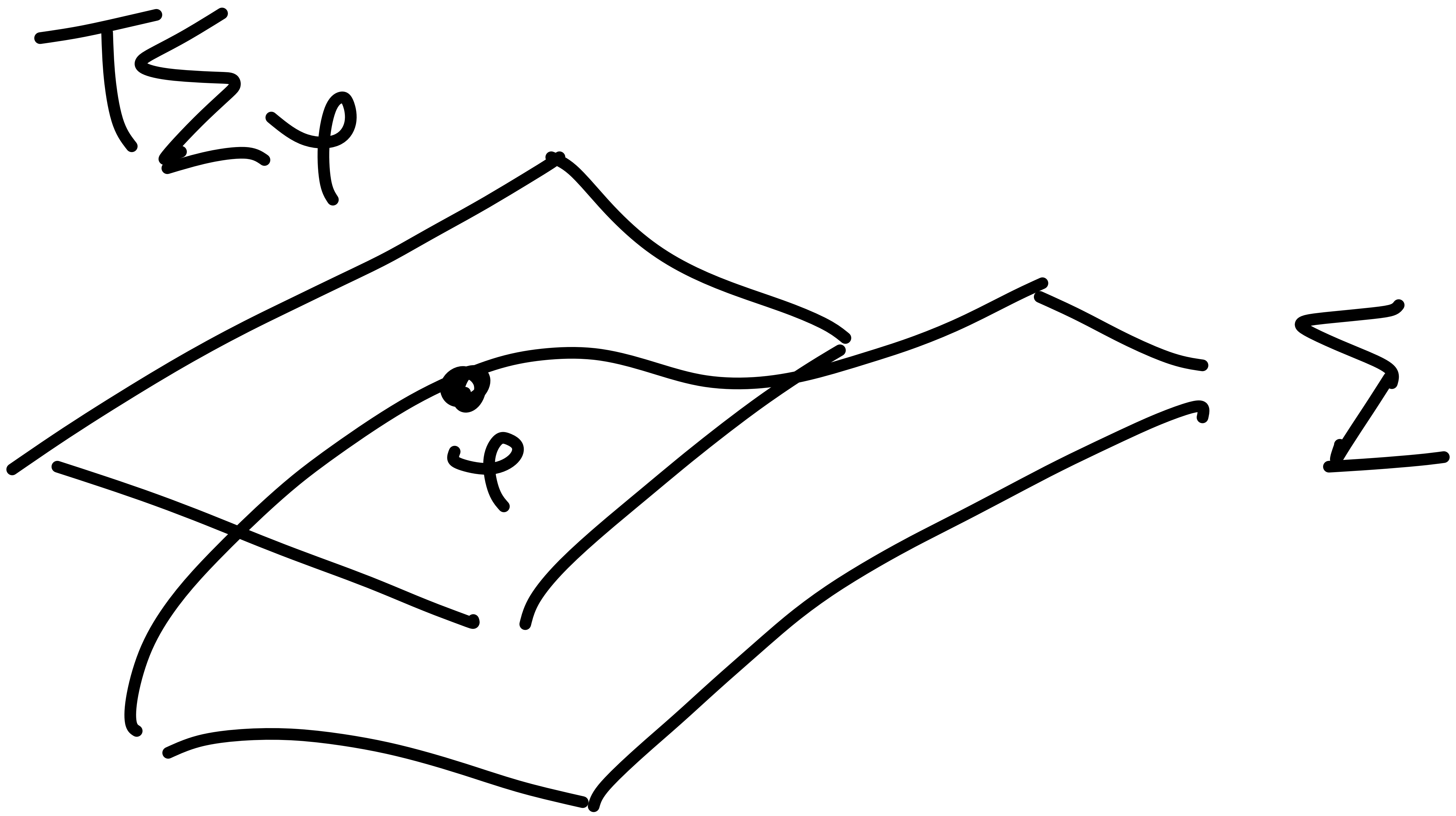

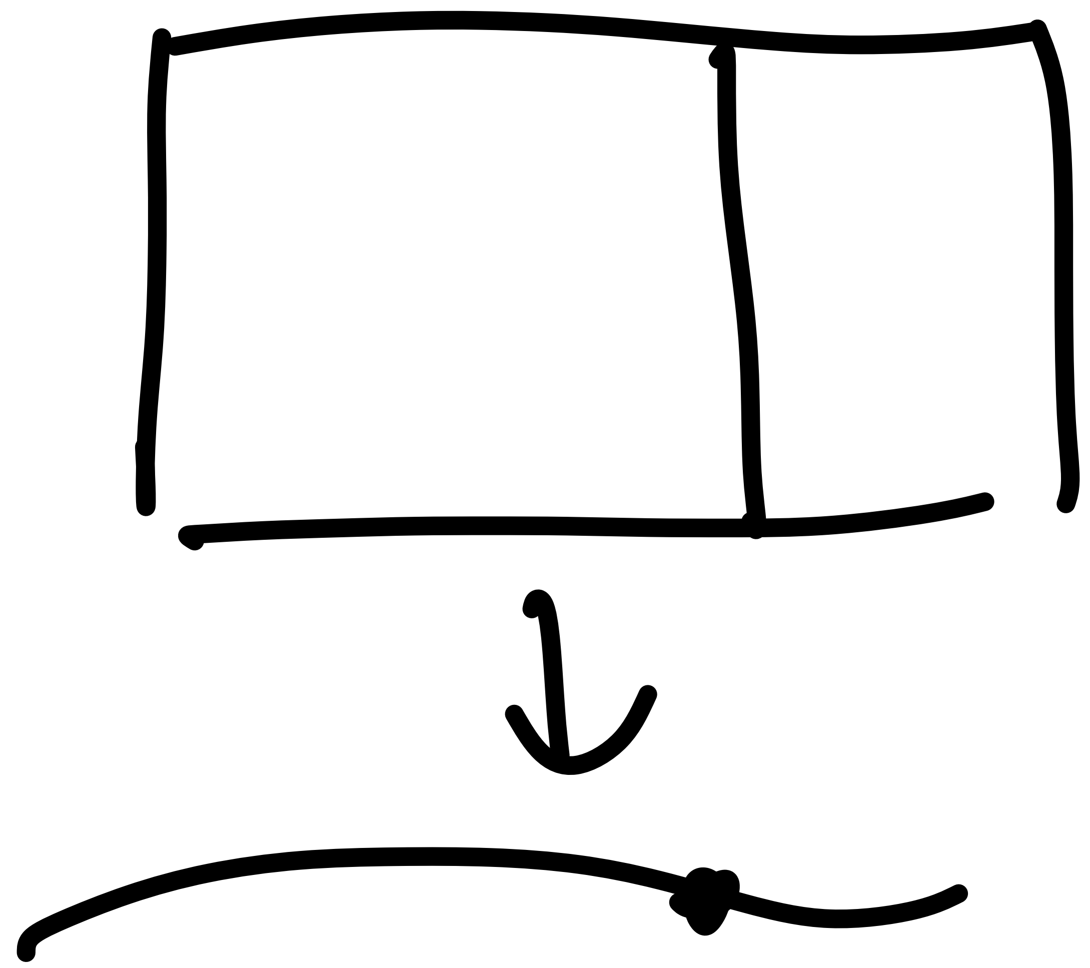

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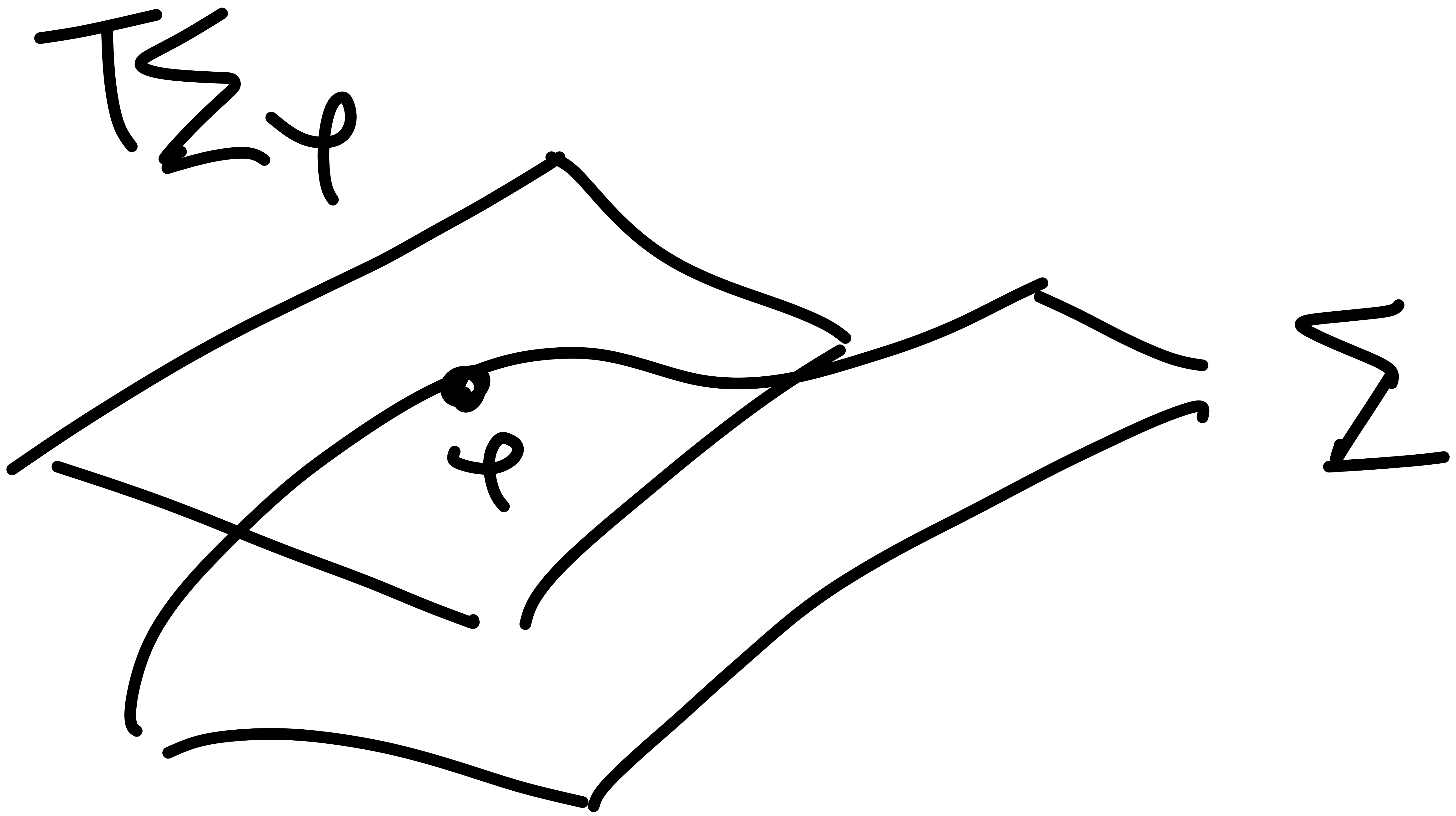

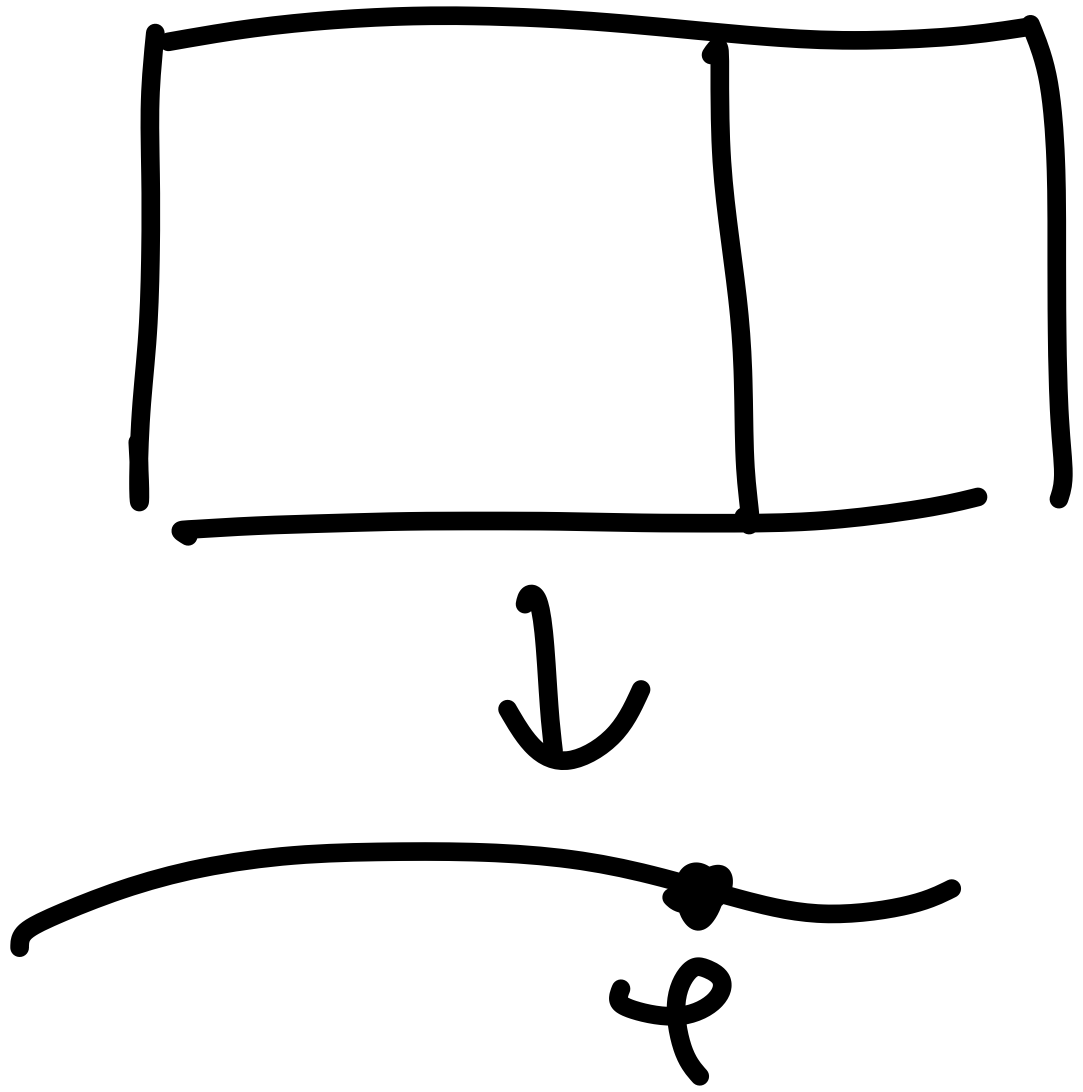

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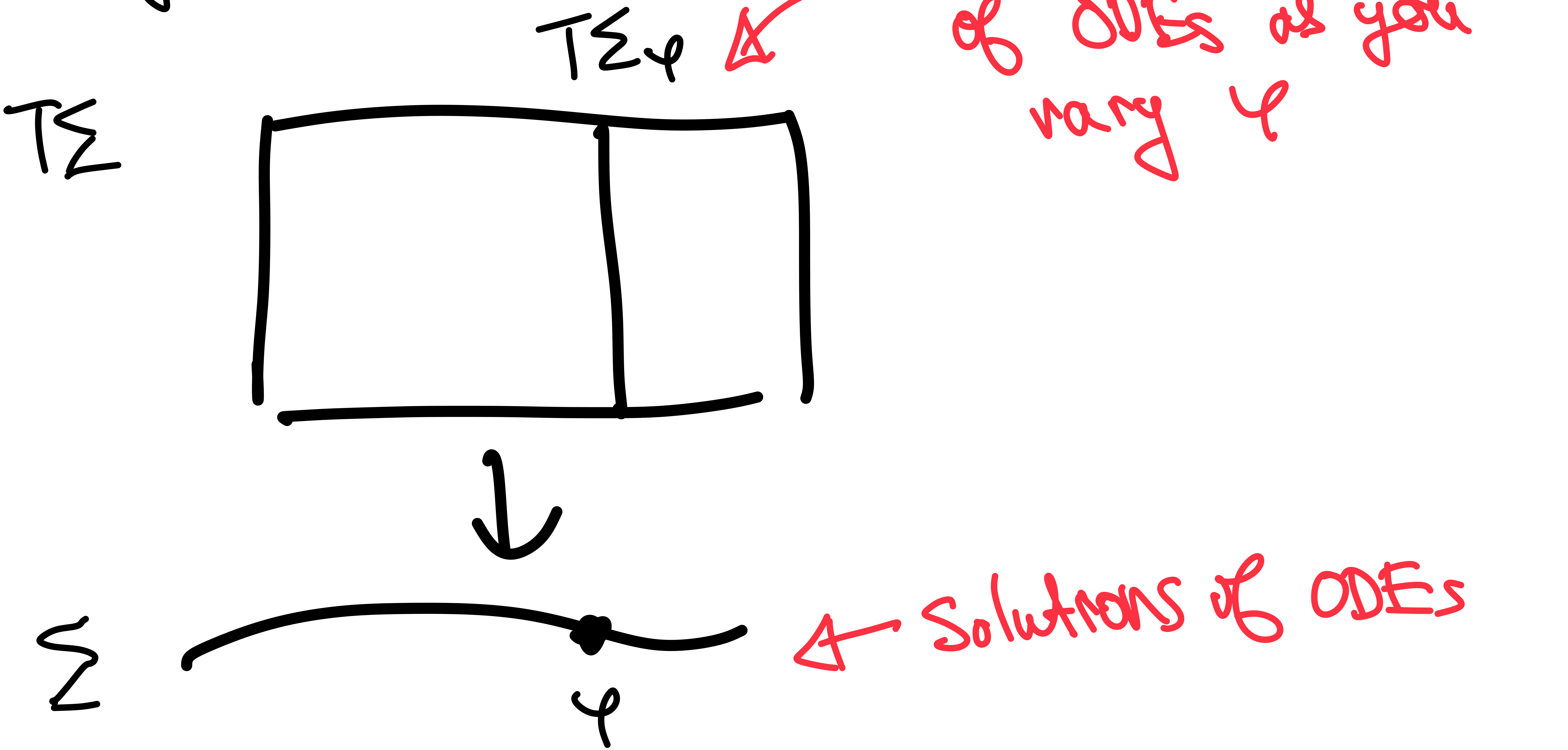

Tangent Bundles

Still have:

$$T\Sigma = \underline{\mathrm{Spec}}\left(\mathrm{Sym}\left(\Omega_{\Sigma/k}\right)\right)$$

Now a 2-algebraic scheme.

THEOREM:

JBC holds for generically reduced components.

Three Key Tools:

1) Generic smoothness of generically reduced components. (Rings are not Noetherian!)

2) Linearization only decreases the Jacobi bound

3) Ritt's proof of linear case.

THEOREM:

JBC holds for generically reduced components.

Proof: (of JBC for Σ_1 generically reduced)

$$\begin{aligned}
\dim(\Sigma_1) &= \dim(U) \\
&= \dim(TU_{\eta_1}) \\
&= \dim((T\Sigma_1)_{\eta_1}) \\
&= \dim(T\Sigma_{\eta_1}) \\
&\leq J(L[u_1,\eta_1], \ldots, L(u_n,\eta_1)) \\
&\leq J(u_1, \ldots, u_n).
\end{aligned}$$

U smooth open (finite type)

tangent space at generic pt computed in different ways

- η_1 generic pt
- equality by generic smoothness

Ritt's theorem for linear ODEs

linearization only decreases Jacobi Bound.

New Results

(joint w/ David Zureick-Brown)

THEOREM:

JBC holds for generically reduced components.

THEOREM:

DC $\Rightarrow$ JBC

New Results

(joint w/ David Zureick-Brown)

THEOREM:

$$DC \Rightarrow JBC$$

THEOREM: JBC holds for generically reduced components.

Strategy of Proof for JBC

THEOREM:
DC $\Rightarrow$ JBC

START WITH: 1) ODE system $\begin{cases} u_1 = 0, \\ u_2 = 0, \\ \vdots \\ u_n = 0, \end{cases}$ $u_i \in K\{x_1, \dots, x_n\}$

2) A component $P_1 \supseteq [u_1, u_2, \dots, u_n]$ minimal prime.

(assumption: $\operatorname{trdeg}_K(\kappa(P_1)) < \infty$)

residue field

WANT TO SHOW: $\operatorname{trdeg}_K(\kappa(P_1)) \leq J(u_1, u_2, \dots, u_n)$

START WITH:

1) $\begin{cases} u_1 = 0, \\ u_2 = 0, \\ \vdots \\ u_n = 0, \end{cases}$

2) $P_1 \supseteq [u_1, u_2, \ldots, u_n]$

$\operatorname{trdeg}_K(k(P_1)) < \infty$

STRATEGY: (induction)

replace

$\begin{cases} v_1 = 0, \\ v_2 = 0, \\ \vdots \\ v_n = 0. \end{cases}$

NEED:

1) $P_1 \supseteq Q_1 \supseteq [v_1, v_2, \ldots, v_n]$

2) $\operatorname{trdeg}_K(k(Q_1)) < \infty$

3) $J(v_1, v_2, \ldots, v_n) \leq J(u_1, u_2, \ldots, u_n)$

WANT TO SHOW: $\operatorname{trdeg}_K(k(P_1)) \leq J(u_1, u_2, \ldots, u_n)$

THEOREM: DC $\Rightarrow$ JBC

QSTN: How can we reduce?

$$\begin{cases} u_1 = 0, \\ u_2 = 0, \\ \vdots \\ u_n = 0, \end{cases} \rightsquigarrow \begin{cases} v_1 = 0, \\ v_2 = 0, \\ \vdots \\ v_n = 0. \end{cases}$$

FIRST ANSWER: Division algorithms.

WANT TO SHOW: $\operatorname{trdeg}_K(R(P_1)) \leq J(u_1, u_2, \dots, u_n)$

THEOREM:

DC $\Rightarrow$ JBC

Division algorithms.

Example:

$$\begin{cases} f = y'' + 2y + 1 \\ g = yy' + 1 \end{cases}$$

leader $l_f = y''$

leader $l_g = y'$

Can perform division: $sf = Q(g) + r$!!!

$s = y^3$, $Q = (1 - yy') + (y^2)\partial \in K\{y\}[\partial]$ ring of diff'l ops

$r = 2y^4 + y^3 - 1$ lower order than g.

WANT TO SHOW: $\operatorname{trdeg}_K(R(P_1)) \leq J(u_1, u_2, \ldots, u_n)$

THEOREM:

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leader $\ell_f = y''$

!!! $\ell_g = y'$

$$sf = Q(g) + r$$

$$s = y^3,$$

WANT TO SHOW: $\text{trdeg}_K(R(P_i)) \leq J(u_1, u_2, \ldots, u_n)$

First part of division algo:

- Take derivative of g to match orders

$$\partial(g) = y''y + (y')^2$$

- Multiply f by y to match leading terms

$$yf = y''y + 2y^2 + y$$

- Subtract

$$yf - \partial(g) = (y')^2 + 2y^2 + y$$

Division algorithms. THEOREM: DC $\Rightarrow$ JBC

Example:

$$\begin{cases} f = y'' + 2y + 1 \\ g = yy' + 1 \end{cases}$$

leader $\ell_f = y''$

!!!

$\ell_g = y'$

$$sf = Q(g) + r$$

$$s = y^3,$$

WANT TO SHOW: $\mathrm{trdeg}_K(R(P_1)) \leq J(u_1, u_2, \ldots, u_n)$

- Multiply f by y to match leading terms

$$\frac{\partial g}{\partial \ell_g} = \frac{\partial g}{\partial y'} = y$$

Defn. $\dfrac{\partial h}{\partial \ell_h} = (\text{separant of } h)$

Division algorithms.

THEOREM:

DC $\Rightarrow$ JBC

$$\begin{cases} f = y'' + 2y + 1 \\ g = yy' + 1 \end{cases} \rightsquigarrow$$

$$sf = Q(g) + r$$

$$s = y^3, \quad \frac{\partial g}{\partial l_g} = \frac{\partial g}{\partial y'} = y$$

UPSHOT: In order to perform division we must multiply by the separant!

Defn. $\dfrac{\partial h}{\partial l_h} = (\text{separant of } h)$

WANT TO SHOW: $\operatorname{trdeg}_K(\mathcal{R}(P_1)) \leq J(u_1, u_2, \dots, u_n)$

Division algorithms.

THEOREM: DC $\Rightarrow$ JBC

QSTN: How can we reduce?

$$\begin{cases} u_1 = 0, \\ u_2 = 0, \\ \vdots \\ u_n = 0, \end{cases} \quad \rightsquigarrow \quad \begin{cases} v_1 = 0, \\ v_2 = 0, \\ \vdots \\ v_n = 0. \end{cases}$$

FIRST ANSWER: Division algorithms.

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$$s_1 = \frac{\partial u_1}{\partial x_1}(r)$$

WLOG: divide u_2 by u_1 in variable x_1

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$s_1 u_2 = Q(u_1) + h$ ← (for $v_2 = h$)

$$s_1 = \frac{\partial u_1}{\partial x_1^{(r)}}$$

WLOG: divide u_2 by u_1 in variable x_1

First Fact: There is a procedure for doing this where

$$J(u_1, \dots, u_n) \geq J(v_1, \dots, v_n)$$

Second Fact: If $s_1 \notin P_1$ then P_1 is a minimal prime of new system.

WANT TO SHOW: $\operatorname{trdeg}_K(R(P_1)) \leq J(u_1, u_2, \dots, u$

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NEED:

1) $P_1 \supseteq Q_1 \supseteq [v_1, v_2, \ldots, v_n]$
2) $\mathrm{trdeg}_K(\mathcal{R}(Q_1)) < \infty$
3) $J(v_1, v_2, \ldots, v_n) \leq J(u_1, u_2, \ldots, u_n)$

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If $S_1 \notin P_1$

First Fact: There is a procedure for doing this where

$$J(u_1, \ldots, u_n) \geq J(v_1, \ldots, v_n)$$

Second Fact: then P_1 is a minimal prime of new system.

WANT TO SHOW: $\operatorname{trdeg}_K(R(P_1)) \leq J(u_1, u_2, \ldots, u$

THEOREM:

DC $\Rightarrow$ JBC

QSTN: How can we reduce?

$$\begin{cases} u_1 = 0, \\ u_2 = 0, \\ \vdots \\ u_n = 0, \end{cases} \quad \xrightarrow{\text{If } s_1 \notin \mathbb{P}_1} \quad \begin{cases} v_1 = 0, \\ v_2 = 0, \\ \vdots \\ v_n = 0. \end{cases}$$

FIRST ANSWER: Division algorithms.

WANT TO SHOW: $\operatorname{trdeg}_K(\mathbb{R}(\mathbb{P}_1)) \leq J(u_1, u_2, \ldots, u_n)$

THEOREM: DC $\Rightarrow$ JBC

QSTN: How can we reduce?

$$\begin{cases} u_1 = 0, \\ u_2 = 0, \\ \vdots \\ u_n = 0, \end{cases} \quad \xrightarrow{\text{If } s_1 \notin \mathfrak{P}_1} \quad \begin{cases} v_1 = 0, \\ v_2 = 0, \\ \vdots \\ v_n = 0. \end{cases}$$

FIRST ANSWER: Division algorithms.

WHAT IF $s_1 \in \mathfrak{P}_1$???

WANT TO SHOW: $\operatorname{trdeg}_K(R(\mathfrak{P}_1)) \leq J(u_1, u_2, \ldots, u_n)$

THEOREM: DC $\Rightarrow$ JBC

QSTN: How can we reduce?

$$\begin{cases} u_1 = 0, \\ u_2 = 0, \\ \vdots \\ u_n = 0, \end{cases} \quad \overset{\text{If } s_1 \in \mathfrak{P}_1}{\rightsquigarrow} \quad \begin{cases} v_1 = 0, \\ v_2 = 0, \\ \vdots \\ v_n = 0, \end{cases}$$

SECOND ANSWER: RiH Pencil Trick

WANT TO SHOW: $\operatorname{trdeg}_K(R(\mathfrak{P}_1)) \leq J(u_1, u_2, \ldots, u_n)$

THEOREM: DC $\Rightarrow$ JBC

THEOREM:

$$DC \Rightarrow JBC$$

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FIRST ANSWER:

Division algorithms.

SECOND ANSWER:

Rith Pencil Trick

[Dimension conj comes in on this side]

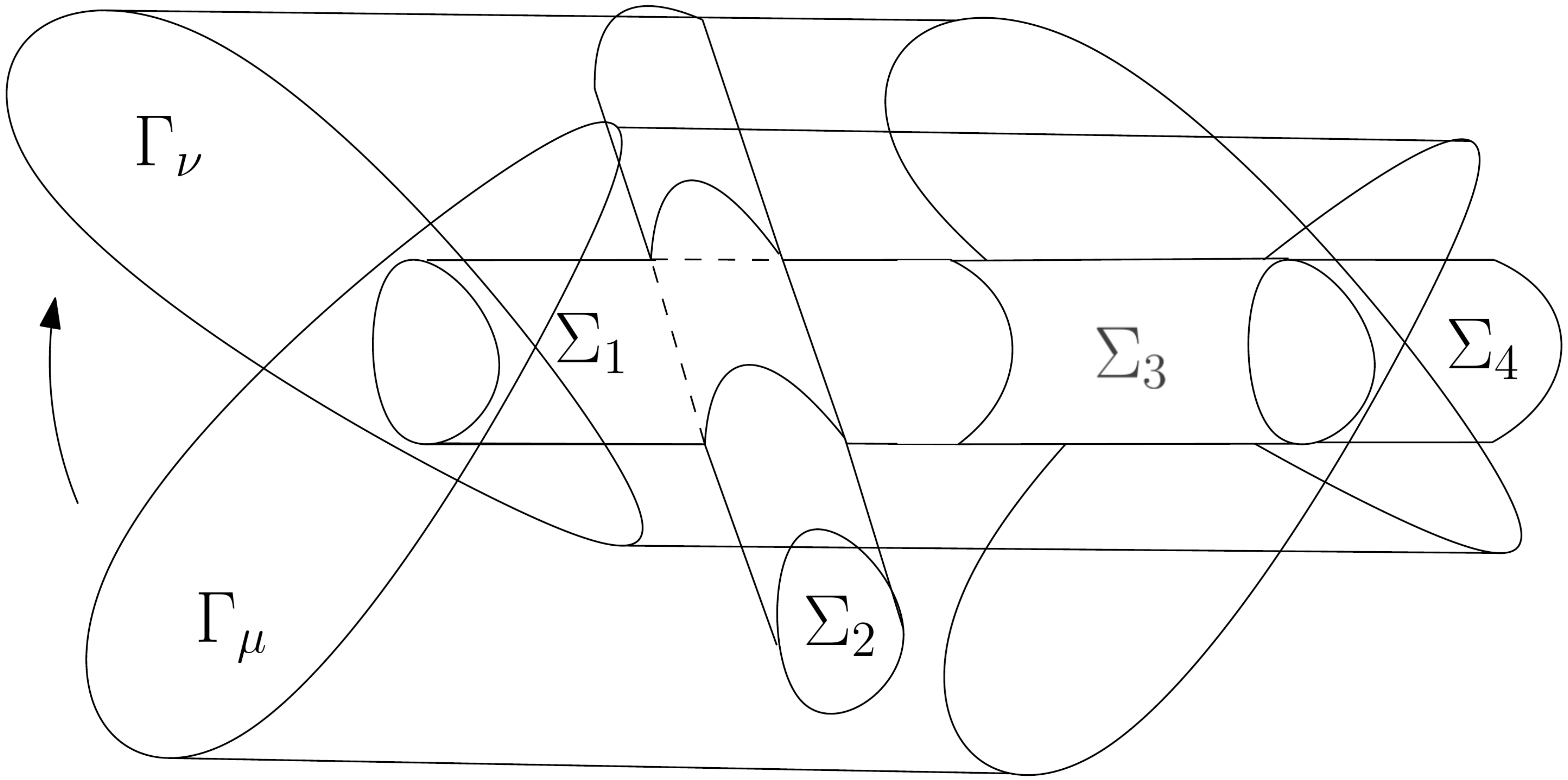
WANT TO SHOW:

$$\operatorname{trdeg}_K(R(P_1)) \leq J(u_1, u_2, \ldots, u_n)$$

QSTN: How can we reduce?

RiH Pencil Trick

If $s_1 \in P_1$



$\Gamma \qquad \Gamma_\mu$

$$\Big\downarrow f$$

$\mathbb{A}^1_{\infty \to \mu}$

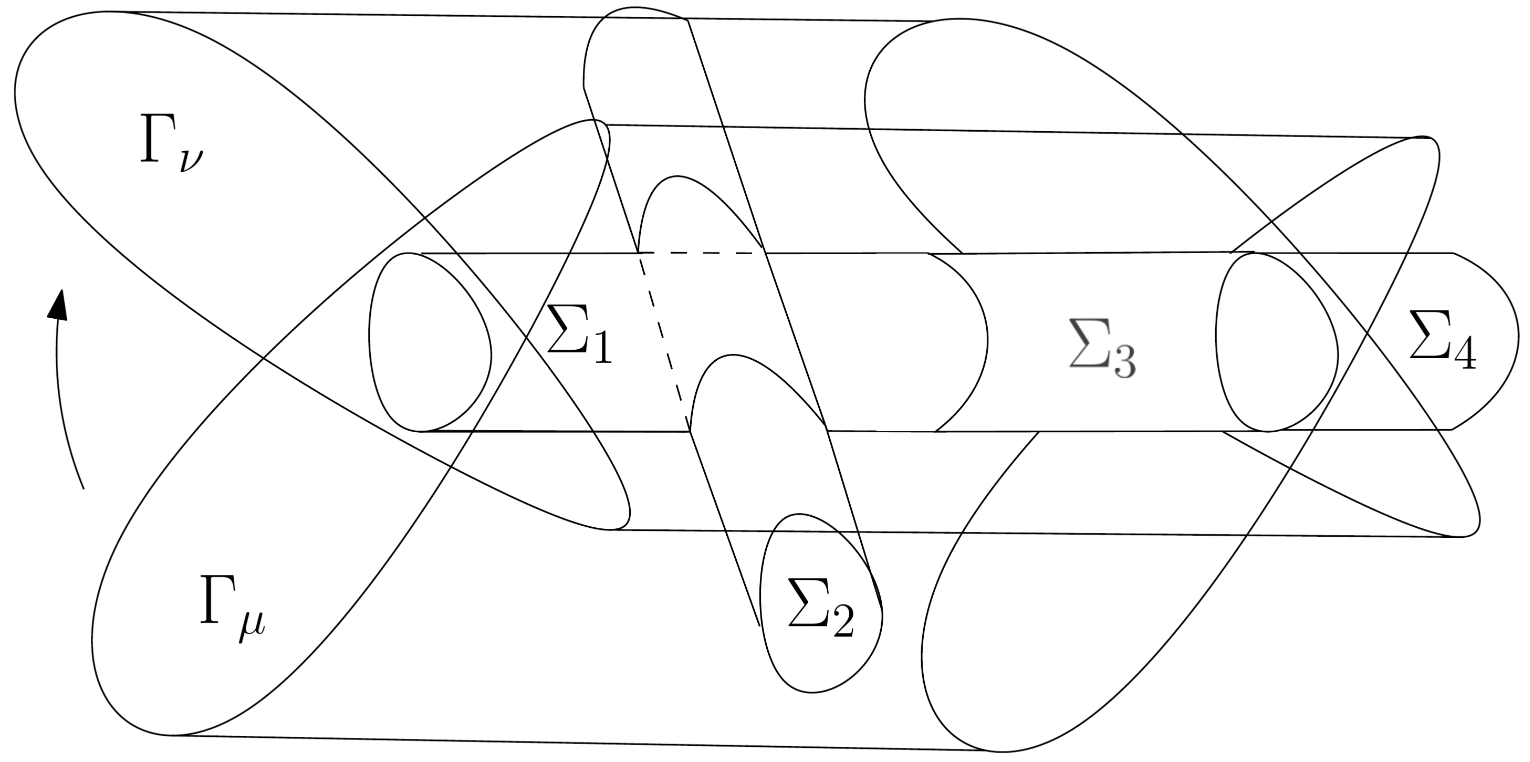
$$\mathbb{A}^n_\infty \supseteq \Gamma_\mu \supseteq BS(\Gamma) \supseteq \Sigma_1$$

$$BS(\Gamma) \supseteq \Sigma$$

RiH Pencil Trick

If $s_1 \in P_1$

$$\Gamma : \begin{cases} v_1 = t_1 + y s_1, \\ v_2 = u_2, \\ \vdots \\ v_n = u_n, \end{cases}$$



$s_1 = \frac{\partial u_1}{\partial \ell}$, $\ell = x_1^{(r)}$, $r = \mathrm{ord}_{x_1}^{\partial}(u_1)$

$$d \cdot u_1 = \ell s_1 + t_1$$

Rift Pencil Trick

If $s_1 \in \mathfrak{P}_1$

$$\Gamma : \begin{cases} v_1 = t_1 + y s_1, \\ v_2 = u_2, \\ \vdots \\ v_n = u_n, \end{cases}$$

$\tilde{\mathfrak{P}}_1 \supseteq [v_1, v_2, \ldots, v_n]$

$\rightsquigarrow$

$\Gamma_1 \subseteq \Gamma$ component

Moving case:

$0 \neq \tilde{\mathfrak{P}}_1 \cap K\{y, x_j\}$

for $1 \leq j \leq n$

Dimension conjecture

Fixed case:

Easy