Automorphism groups of toroidal horospherical varieties

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Motivation

We work over the field $\mathbb{C}$ of complex numbers and mainly consider smooth complete varieties.

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Theorem (Matsushima 1957)

If a Fano manifold X admits a Kähler–Einstein metric, then $\mathrm{Aut}^0(X)$ is reductive.

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Determine the reductivity of $\mathrm{Aut}^0(X)$ *for a given variety* X.

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- For a given toric variety X, the reductivity of $\mathrm{Aut}^0(X)$ can be easily computed in terms of Demazure roots.

Goal

Generalize the notion of Demazure roots for toroidal horospherical varieties.

Outline

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Automorphism groups of complete varieties

Let X be a complete variety. Then $\mathrm{Aut}(X)$ is a smooth group scheme, locally of finite type, so we have

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By [Barsotti 1955, Chevalley 1960], we have

$$1 \to \mathrm{Aut}^0_{lin}(X) \to \mathrm{Aut}^0(X) \stackrel{Alb}{\to} \mathrm{Aut}^0_{ab}(X) \to 1$$

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By Levi decompostion,

$$\mathrm{Aut}^0(X) = R_u(\mathrm{Aut}^0(X)) \rtimes L$$

where $R_u(\mathrm{Aut}^0(X))$ is the unipotent radical of $\mathrm{Aut}^0(X)$ and L is a reductive group.

Outline

Demazure roots of toric varieties

Let F be a smooth complete S-toric variety where $S \cong (\mathbb{C}^*)^{\dim F}$ is the big torus.

Definition

Let F be a toric variety and Σ be the corresponding fan in $N_{\mathbb{R}}$. Let

$$\mathcal{R}(F) = \{m \in M \mid \exists \rho \in \Sigma(1) : \langle m, v_\rho \rangle = -1 \ \& \ \langle m, v_{\rho'} \rangle \geq 0 \ \forall \rho' \in \Sigma(1) \setminus \{\rho\}\}$$

where v_ρ and $v_{\rho'}$ denote the primitive generator of the ray ρ and ρ', resp. An element of $\mathcal{R}(F)$ is called a Demazure root for Σ.

- $\mathcal{R}(F)$ depends only on $\Sigma(1)$.

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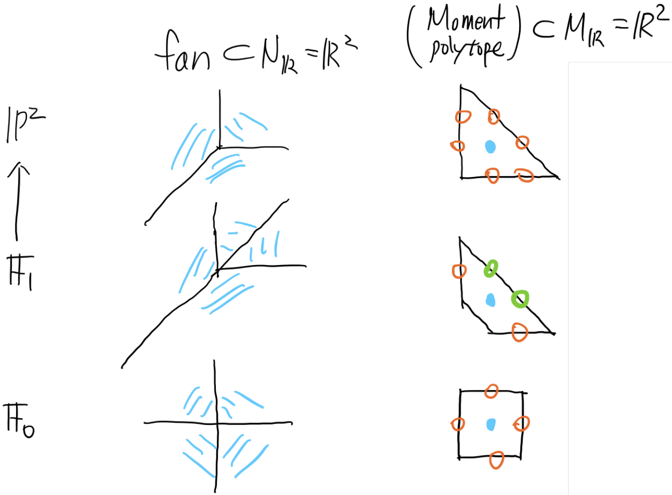
Definition

- $\mathcal{S}(F) = \{m \in \mathcal{R}(F) : -m \in \mathcal{R}(F)\}$
- $\mathcal{U}(F) = \mathcal{R}(F) \setminus \mathcal{S}(F)$

Elements of $\mathcal{S}(F)$ and $\mathcal{U}(F)$ are called semisimple and unipotent, respectively.

Example



fan ⊂ $N_{\mathbb{R}} = \mathbb{R}^2$
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$\mathbb{P}^2$
$\mathbb{F}_1$
$\mathbb{F}_0$

Normalized $\mathbb{G}_a$-actions

Definition

Let L be a linear algebraic group, $\omega \in \mathrm{Hom}(L, \mathbb{C}^*)$ be a character of L, and Y be an L-variety.

- A $\mathbb{G}_a$-action $A : \mathbb{G}_a \times Y \to Y$ is L-normalized with weight ω if

$$l.A(s, l^{-1}.y) = A(\omega(l)s, y)$$

  for every $l \in L, s \in \mathbb{G}_a, y \in Y$.
- A subgroup H of L is a $\mathbb{G}_a$-subgroup of L if $H \cong \mathbb{G}_a$.
- An L-root subgroup is the $\mathbb{G}_a$-subgroup of L associated to the $\mathbb{G}_a$-action A.

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  1. $\mathcal{R}(F)$
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- Let U_m be the S-root subgroup of $\mathrm{Aut}^0(X)$ corresponding to $m \in \mathcal{R}(F)$.

Demazure roots for toric varieties

Theorem (Demazure)

Let F be a complete S-toric variety.

1. *The S-root subgroups U_m for $m \in \mathcal{U}(F)$ generate the unipotent radical $R^u(\mathrm{Aut}^0(F))$, and S and U_m for $m \in \mathcal{S}(F)$ generate a Levi subgroup of $\mathrm{Aut}^0(F)$.*

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$$\dim(\mathrm{Aut}^0(F)) = \dim(S) + |\mathcal{R}(F)| = \dim(S) + |\mathcal{S}(F)| + |\mathcal{U}(F)|.$$

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Theorem (Demazure)

Let F be a complete S-toric variety with fan Σ. As S-modules,

$$\mathrm{Lie}(\mathrm{Aut}^0(F)) = \mathrm{Lie}(S) \oplus \bigoplus_{m \in \mathcal{R}(F)} \mathrm{Lie}(U_m).$$

Demazure roots of toric varieties: examples

fan $\subset N_{\mathbb{R}} = \mathbb{R}^2$ $\qquad$ (Moment polytope) $\subset M_{\mathbb{R}} = \mathbb{R}^2$

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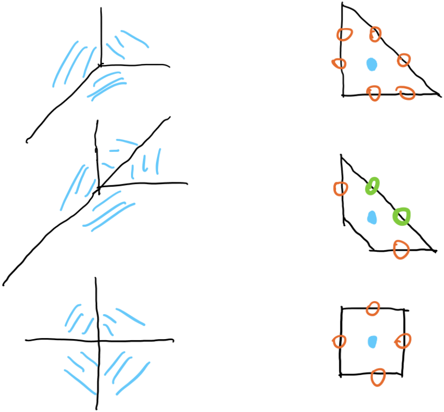

$\dim \mathrm{Aut}^0(\mathbb{P}^2) = 2+6 = 8$

$\dim \mathrm{Aut}^0(\mathbb{F}_1) = 2+4 = 6$

$\dim \mathrm{Aut}^0(\mathbb{F}_0) = 2+4 = 6$

Recall $\mathrm{Aut}(\mathbb{F}_n) \simeq \mathbb{G}_a^{n+1} \rtimes (GL_2/\mu_n)$ for $n > 0$

$\mathrm{Aut}(\mathbb{F}_0) \simeq (PGL_2 \times PGL_2) \rtimes \{id, \sigma\}$

Today's goal

Problem

- *Study the structure of* $\mathrm{Aut}^0(X)$ *for a toroidal horospherical variety* X.
- *In particular, find the reductivity criterion for* $\mathrm{Aut}^0(X)$.

Outline

Horospherical varieties

Definition

Let G be a connected reductive group.

1. A closed subgroup H of G is horospherical if H contains $R^u(B)$ for a Borel subgroup B of G.
2. A normal G-variety X is horospherical if X has a dense open subset that is isomorphic to G/H for some horospherical subgroup H of G.

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1. (Rational homogeneous spaces) Let P be a parabolic subgroup of G, i.e., it is closed and contains a Borel subgroup. Thus P is horospherical. Then G/P is a horospherical variety.
2. (Toric varieties) Let $G = (\mathbb{C}^*)^n$ be a torus. Take H to be a trivial subgroup of G. Then H is horospherical. Thus any toric variety is horospherical.

Principal torus bundle structure on G/P

Theorem (Pasquier)

Let H be a horospherical subgroup of G and $P = N_G(H)$. Then P is a parabolic subgroup of G and P/H is a torus. Moreover, G/H has a principal torus bundle structure on the rational homogeneous space G/P

$$G/H \to G/P$$

where the torus fiber is isomorphic to $P/H =: S$.

Toroidal horospherical varieties

Definition

A horospherical variety X is toroidal if every B-stable divisor is G-stable.

- A toroidal horospherical variety X admits a morphism $\pi : X \to G/P$ extending the natural morphism $G/H \to G/P$.
- π is a toric bundle over G/P whose S-toric fiber is denoted by F.

Proposition

For every G-horospherical variety X, there exists a G-equivariant birational morphism $\tilde{X} \to X$ from a toroidal horospherical variety $\tilde{X}$.

Example

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. We consider two toroidal horospherical structures.

1. X is a $(\mathbb{C}^*)^2$-horospherical variety, that is, a toric variety.
2. X is a $(SL_2 \times \mathbb{G}_m)$-horospherical variety.

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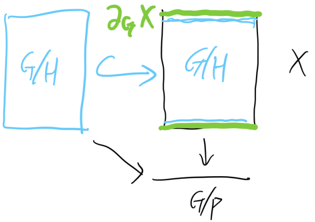
- Then $B = B^- = B_{SL_2} \times \mathbb{G}_m$ and $R^u(B) = U_{SL_2} \times \{1\}$.
- Take $H = B_{SL_2} \times \{1\}$.
- $P = B_{SL_2} \times \mathbb{G}_m$.
- So $G/P \cong \mathbb{P}^1$ and $G/H \cong \mathbb{P}^1 \times \mathbb{G}_m$.



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- How to extend the Demazure roots for F to that for X? (surjectivity)

Demazure roots for horospherical varieties

Definition (-Proposition)

$$\mathcal{R}_G^+(X) = \{m \in \mathcal{R}_S(F) : m \text{ is } B^+\text{-dominant}\}$$
$$(= \{m \in \mathcal{R}_S(F) : \langle m, \epsilon^+(\mathcal{D}^+)\rangle \geq 0\})$$

Elements of $\mathcal{R}_G^+(X)$ are called Demazure roots (B^+-roots) of X.

- $\mathcal{D}^+$ is the opposite Borel subgroup of G.

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Theorem (Barban–H–Kwon)

For each Demazure root $m \in \mathcal{R}_S(F)$, the following are equivalent

1. $m \in \mathcal{R}_G^+(X)$
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- $\epsilon^+ : \mathcal{D}^+ \to N_{G/H}$ is the color map defined by $\langle m, \epsilon^+(\mathcal{D}^+)\rangle = \nu_D(f_m)$ for $f_m \in \mathbb{C}(G/H)^{(B)}_m := \{f \in \mathbb{C}(G/H) : b.f = \xi(b)f \ \forall b \in B\}$.

Reductivity

Definition

- $\mathcal{S}_G^+(X) = \{m \in \mathcal{R}_G^+(X) \colon -m \in \mathcal{R}_G^+(X)\}$
- $\mathcal{U}_G^+(X) = \mathcal{R}_G^+(X) \setminus \mathcal{S}_G^+(X)$

Elements of $\mathcal{S}_G^+(X)$ and $\mathcal{U}_G^+(X)$ are called semisimple and unipotent, respectively.

Theorem (Barban–H–Kwon)

$\mathrm{Aut}^0(X)$ *is reductive if and only if* $\mathcal{U}_G^+(X) = \emptyset$.

Decomposition

For $m \in \mathcal{R}_G^+(X)$,
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- *As G-modules*

$$\mathrm{Lie}(\mathrm{Aut}^0(X)) = \mathrm{Lie}(\mathrm{Aut}^0(G/P)) \oplus \mathrm{Lie}(\mathrm{Aut}_G(X)) \oplus \bigoplus_{m \in \mathcal{R}_G^+(X)} V(m).$$

- *In particular*

$$\dim \mathrm{Aut}^0(X) = \dim \mathrm{Aut}^0(G/P) + \dim S + |\mathcal{S}_G^+(X)| + \sum_{m \in \mathcal{U}_G^+(X)} \dim V(m).$$

- $\mathrm{Aut}^0(X)$ *is generated by* $\mathrm{Aut}^0(X, \partial_G X)$ *and* U_m^+*, for* $m \in \mathcal{R}_G^+(X)$.

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- $G \to \mathrm{Aut}^0(G/P)$ is surjective in most cases.

Historical remarks

- The structure of $\mathrm{Aut}^0(X)$ for a smooth complete spherical variety X is studied in [Bien–Brion96], [Brion07] and [Pezzini09], which was used in our proof.
- $\mathrm{Aut}^0(X)$ for an (quasi-)affine spherical variety X is systematically studied by Avdeev and Zhgoon. In fact, they informed us that they obtained similar results as ours using their preprint arXiv:2312.03377 along with the complete description of the Lie algebra structure, which will be reflected in their manuscript later.

Example

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. X is a $(SL_2 \times \mathbb{G}_m)$-horospherical variety.

- Take $G = SL_2 \times \mathbb{G}_m$.
- Then $B = B^- = B_{SL_2} \times \mathbb{G}_m$ and $R^u(B) = U_{SL_2} \times \{1\}$.
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- $B^+ = B^{opp} = B_{SL_2}^{opp} \times \mathbb{G}_m$.

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- $D^+ = \{\infty\} \times \mathbb{P}^1$ is the unique B^+-color in $\mathbb{P}^1 \times \mathbb{P}^1$.

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&= 3 + 1 + 2 + 0 \\
&= 6.
\end{aligned}$$

Outline

$\mathbb{P}^n$-bundles over G/P

From now on, Y is a rational homogeneous space, and $L_1, \ldots, L_k$ are line bundles on Y. ($k \geq 2$)

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Let $X = \mathbb{P}_Y(L_1 \oplus \cdots \oplus L_k)$. Let $G = (\mathbb{C}^\times)^k \times \widetilde{\mathrm{Aut}^0(Y)}$ where $\widetilde{\mathrm{Aut}^0(Y)}$ denotes the universal cover of $\mathrm{Aut}^0(Y)$.
Then X is a toroidal horospherical G-variety.

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Reductivity using Demazure roots

Let P be the parabolic subgroup of G containing B^- such that $Y \simeq G/P$.
For each i, let χ_i be the character of P such that $L_i = G \times^P \mathbb{C}_{\chi_i}$.

Theorem (Barban–H–Kwon)

Let $X = \mathbb{P}_Y(L_1 \oplus \cdots \oplus L_k)$. *Then*

- $\mathcal{R}_G^+(X)$ *consists of* $\chi_i - \chi_j$ *(*$i \neq j$*) such that* $L_i - L_j$ *is nef on* Y.
- $\mathcal{S}_G^+(X)$ *consists of* $\chi_i - \chi_j$ *(*$i \neq j$*) such that* $L_i \simeq L_j$ *as line bundles on* Y.

Theorem (Barban–H–Kwon)

Let $X = \mathbb{P}_Y(L_1 \oplus \cdots \oplus L_k)$. *Then* $\mathrm{Aut}^0(X)$ *is reductive if and only if for any* i, j *with* $i \neq j$ *such that* $L_i \not\simeq L_j$, $L_i - L_j$ *is not nef.*

K-stability of $\mathbb{P}^1$-bundles

Theorem (Ziquan Zhuang 2020)

Let X_1 and X_2 be two Fano varieties such that $X = X_1 \times X_2$. Then X is K-semistable (resp. K-polystable, K-stable, uniformly K-stable) if and only if X_1 and X_2 are both K-semistable (resp. K-polystable, K-stable, uniformly K-stable).

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Question

Study the K-stability of projective space bundles over a Fano variety.

Theorem (Kewei Zhang–Chuyu Zhou 2022)

Let Y be a Fano variety of dimension n and Fano index ≥ 2. Take $L = -\frac{1}{r}K_Y$ for a rational number $r > 1$. Then $\mathbb{P}_Y(\mathcal{O}_Y \oplus L^\vee)$ is K-unstable.

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Assume $k = 2$ and $L_1 = \mathcal{O}_Y$. Let $L = L_2$.

Corollary (Barban–H–Kwon)

Let $X = \mathbb{P}_Y(\mathcal{O}_Y \oplus L)$. *Then*

- $\mathrm{Aut}^0(X)$ *is reductive if and only if either* L *is trivial, or neither* L *nor* $-L$ *is nef on* Y.
- *In particular, if* Y *has Picard number* 1*,* $\mathrm{Aut}^0(X)$ *is reductive if and only if* $X \simeq Y \times \mathbb{P}^1$.

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Example

Let $Y = (\mathbb{P}^1)^n$, and consider the $\mathbb{P}^1$-bundle $X = \mathbb{P}(\mathcal{O}_Y \oplus L)$, where $L = \mathcal{O}_Y(a_1, \ldots, a_n)$, with $a_i \in \mathbb{Z}$ for every $i = 1, \ldots, n$.

- X is Fano if and only if $a_i \in \{-1, 0, 1\}$ for every $i = 1, \ldots, n$.
- $\mathrm{Aut}^0(X)$ is reductive if and only if either $a_1 = \ldots = a_n = 0$, or there exists $i \neq j$ such that $a_i a_j < 0$, for $i, j = 1, \ldots, n$.

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Thank you.