# Homological sieve and Manin's conjecture

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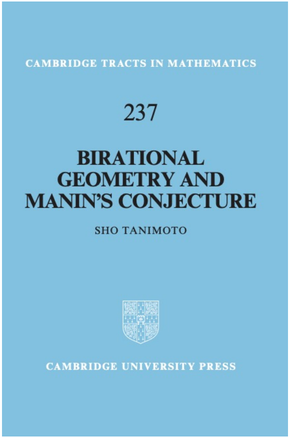
# My book!!



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237

BIRATIONAL
GEOMETRY AND
MANIN'S CONJECTURE

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# Manin’s conjecture

## I. Manin’s conjecture

# Height functions

$\mathbb{P}^n/\mathbb{Q}$ : the $n$-dimensional projective space over $\mathbb{Q}$.

A height function $\mathsf{H} : \mathbb{P}^n(\mathbb{Q}) \to \mathbb{R}_{>0}$ is defined by

$$\mathsf{H}(x_0 : \cdots : x_n) = \max\{|x_0|, \cdots, |x_n|\},$$

where $x_i \in \mathbb{Z}$ and $\gcd_i(x_i) = 1$.

Note that

$$\{P \in \mathbb{P}^n(\mathbb{Q}) \mid \mathsf{H}(P) \leq T\}$$

is a finite set.

## Height functions

$k$: a number field,
$X$: a projective variety defined over $k$,
$L$: a Cartier divisor on $X$.

$\Longrightarrow$ a height function $\mathrm{H}_L : X(k) \to \mathbb{R}_{>0}$ associated to a triple $(k, X, L)$.

When $L$ is ample, the set of rational points of bounded height is finite:

$$\mathrm{N}(Q, L, T) := \#\{P \in Q \mid \mathrm{H}_L(P) \leq T\} < \infty$$

where $Q \subset X(k)$ is any subset.

Arithmetic geometers seek the asymptotic formula of this counting function as $T \to \infty$ for an appropriate choice of $Q$.

# Smooth Fano varieties

$k$: a field

$X$: a smooth projective variety defined over $k$

$X$ is a Fano variety if $-K_X$ is ample.

Examples:

- cubic hypersurfaces in $\mathbb{P}^n$ with $n \geq 3$;
- complete intersections of two quadrics in $\mathbb{P}^n$ with $n \geq 4$.

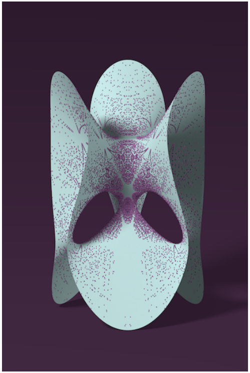
# Del Pezzo surfaces

Del Pezzo surfaces
are 2-dimensional Fano varieties

$S$: a del Pezzo surface over $k$
$e = (-K_S)^2$: the degree of $S$
$1 \leq e \leq 9$

**Examples:**

- $e = 9$ if and only if $S \cong \mathbb{P}^2$ assuming $\overline{k} = k$
- $e = 3$ if and only if $S$ is a smooth cubic surface in $\mathbb{P}^3$
- $e = 4$ if and only if $S$ is a complete intersection of two quadrics in $\mathbb{P}^4$



[Credit: U. Derenthal, O. Labs]

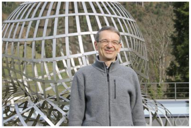
## Manin's conjecture for Fano varieties

**Conjecture (Manin's conjecture, due to Batyrev–Manin–Peyre–Tschinkel)**

*$X$: a smooth Fano variety defined over a number field $k$ s. t. $X(k)$ is not thin. Then there exists a thin set $Z \subset X(k)$ such that*

$$N(X(k)\setminus Z, -K_X, T) \sim \alpha(\mathrm{Nef}_1(X))\beta(X)\tau_{-K_X}(X)T(\log T)^{\rho(X)-1},$$

*as $T \to \infty$ where $\rho(X)$ is the Picard rank of $X$, $\alpha(\mathrm{Nef}_1(X))$ is the alpha constant, $\beta(X)$ is the size of the Brauer group modulo constants, and $\tau_{-K_X}(X)$ is the Tamagawa number introduced by Peyre and Batyrev–Tschinkel.*




## Manin's conjecture for Fano varieties

- A thin set $Z$ is a subset of any finite union $\cup_i f_i(Y_i(k))$ where $f_i : Y_i \to X$ is generically finite onto the image, but not birational to $X$.
- Originally it was expected that $Z$ should be $V(k)$ where $V \subsetneq X$ is proper closed.
- Batyrev–Tschinkel first came up with a counter example to the closed set version of Manin's conjecture in higher dimension.
- Peyre was the first to suggest to remove a thin set.
- A thin set $Z$ is necessary even in dimension 2 due to Gao.

How about del Pezzo surfaces?

## The status of the conjecture for smooth del Pezzo surfaces

1. Del Pezzo surfaces of degree $\geq 6$ are toric, so this is done by Batyrev–Tschinkel;
2. Split del Pezzo surfaces of degree 5 over $\mathbb{Q}$ by de la Bretèche
   Split dP5 over arbitrary number fields by Bernert–Derenthal,
   Some non-split cases over $\mathbb{Q}$ by Heath-Brown–Loughran;
3. An example of a degree 4 del Pezzo surface by de la Bretèche–Browning;
4. Upper bounds of correct magnitude for degree 4 del Pezzo surfaces with conic bundle structures by Browning–Sofos;
5. Lower bounds of correct magnitude for del Pezzo surfaces with conic bundle structures by Frei–Loughran–Sofos.

[3, 4, 5] are based on conic bundle structures!!

Main question: How about over $\mathbb{F}_q(t)$?

# The space of rational curves on a del Pezzo surface

## II. The space of rational curves on a del Pezzo surface

# Rational points vs rational curves

$k$: a field

$K = k(t)$: the function field of one variable over $k$

$X$: a projective variety over $k$

By valuative criterion, we have a natural correspondence

$$\begin{aligned} x \in X(K) &\iff \text{a section } \sigma : \mathbb{P}^1 \to X \times \mathbb{P}^1 \text{ over } \mathbb{P}^1 \\ &\iff \text{a rational curve } s : \mathbb{P}^1 \to X \end{aligned}$$

Thus we need to understand the set of rational curves which forms the moduli space!

## The space of rational curves on a Fano variety

$k$: a field
$X$: a Fano variety over $k$
$\alpha$: a numerical class of 1-cycles

$s : \mathbb{P}^1 \to X$: a rational curve on $X$

The space of rational curves is

$$\mathrm{Mor}(\mathbb{P}^1, X, \alpha)$$

the morphism scheme parametrizing $s : \mathbb{P}^1 \to X$ such that $s_*[\mathbb{P}^1] = \alpha$.

## Manin's conjecture over $\mathbb{F}_q$

Over $\mathbb{F}_q$, our counting function is defined by

$$N(\mathbb{P}^1, X, -K_X, d) = \sum_{\alpha \in \mathrm{Nef}_1(X)_{\mathbb{Z}}, -K_X.\alpha \leq d} \#\mathrm{Mor}(\mathbb{P}^1, X, \alpha)(\mathbb{F}_q).$$

It is counting the number of $\mathbb{F}_q$-nef rational curves of degree $\leq d$.

Manin's conjecture predicts

$$N(\mathbb{P}^1, X, -K_X, d) \sim (1-q^{-1})^{-1}\alpha(\mathrm{Nef}_1(X))\beta(X)\tau_{-K_X}(X)q^d d^{\rho(X)-1},$$

as $d \to \infty$.

## Batyrev's heuristics

In his lecture in 1988 in Berlin,
Batyrev gave a heuristics for this asymptotic formula.

His heuristics is based on the following three assumptions:

1. $\exists U$: a Zariski open $U$ s.t. for any component $M \subset \mathrm{Mor}_U(\mathbb{P}^1, X)$, $M$ is dominant and we have
$$\dim M = -K_X.C + \dim X \quad (C \in M)$$
2. for any $\alpha \in \mathrm{Nef}_1(X)_{\mathbb{Z}}$, $\mathrm{Mor}_U(\mathbb{P}^1, X, \alpha)$ is geometrically irreducible, and;
3. (Ellenberg–Venkatesh) Cohomology of $\mathrm{Mor}_U(\mathbb{P}^1, X, \alpha)$ satisfies certain homological stability so that using GL-trace formula one can prove
$$\#\mathrm{Mor}_U(\mathbb{P}^1, X, \beta)(\mathbb{F}_q) \sim cq^{\dim \mathrm{Mor}_U(\mathbb{P}^1, X, \beta)}.$$

## Batyrev's heuristics

Under these assumptions, we can compute

$$\begin{aligned} N(\mathbb{P}^1, X, -K_X, d) &= \sum_{\alpha \in \mathrm{Nef}_1(X)_{\mathbb{Z}} \ -K_X.\alpha \leq d} \#\mathrm{Mor}_U(\mathbb{P}^1, X, \alpha)(\mathbb{F}_q) \\ &\sim \sum_{\alpha \in \mathrm{Nef}_1(X)_{\mathbb{Z}} \ -K_X.\alpha \leq d} cq^{-K_X.\alpha + \dim X} \\ &\sim (1 - q^{-1})^{-1} \alpha(\mathrm{Nef}_1(X)) cq^d d^{\rho(X)-1} \end{aligned}$$

Are these assumptions correct?

1. Correct in char 0. (Lehmann-T, '19)
2. Not correct but there is an alternative, e.g., geometric Manin's conjecture
3. Unknown in general, but we exhibit examples of this kind in this talk.

# Main Theorem

$S$: a split smooth quartic del Pezzo surface over $k = \mathbb{F}_q$

$\ell$: a non-negative, rational, homogeneous, continuous, and piecewise linear function on the nef cone $\mathrm{Nef}_1(S)$.

$\epsilon > 0$: a small rational number.

$\mathrm{Nef}_1(S)_{\ell,\epsilon}$ : the closure of the set of non-zero classes $\alpha \in \mathrm{Nef}_1(S)$ such that $\ell(\alpha)/(-K_S.\alpha) \geq \epsilon$.

$d$: a positive integer,
$N(\mathbb{P}^1, S, -K_S, d, \epsilon)$: the number of $\mathbb{F}_q$-morphisms $s : \mathbb{P}^1 \to S$ of anticanonical degree $\leq d$ such that the class of the image is contained in $\mathrm{Nef}_1(S)_{\ell,\epsilon}$.

# Main Theorem

**Theorem (Das–Lehmann–T–Tosteson, '25)**

*$C = 2^{32}$, $0 < \epsilon \ll 1$, $q$ with $q^{\epsilon} > C$.*
*$S$: a split smooth quartic del Pezzo surface over $\mathbb{F}_q$*

*Then there exists $\ell$ on $\mathrm{Nef}_1(S)$ which is positive on a dense open cone $U \subset \mathrm{Nef}_1(S)$*

$$N(\mathbb{P}^1, S, -K_S, d, \epsilon) \sim_{d\to\infty} (1-q^{-1})^{-1}\alpha(\mathrm{Nef}_1(S)_{\ell,\epsilon})\tau_{-K_S}(S)q^d d^5,$$

*where $\alpha(\mathrm{Nef}_1(S)_{\ell,\epsilon})$ is the alpha constant and $\tau_{-K_S}(S)$ is the Tamagawa number.*
*More precisely it is given as the Euler product*

$$\tau_{-K_S}(S) = q^2\left(1-q^{-1}\right)^{-6}\left(\prod_{c\in|\mathbb{P}^1|}\left(1-q^{-|c|}\right)^6\frac{\#S(\mathbb{F}_{q^{|c|}})}{q^{2|c|}}\right).$$

# Main Theorem

**Remark**

*This gives a lower bound of correct magnitude for the counting function for a split quartic del Pezzo surface.*

**Remark**

*Our method has implications to other split del Pezzo surfaces too. In particular this includes*

- *Peyre's all height approach to Manin's conj. for split dP surfaces of degree* $\geq 5$
- *Lower bounds of correct magnitude for split del Pezzo surfaces of degree* $\leq 3$.

## Fibrations induced by a conic bundle

$\alpha$: a nef class
$\exists$ a birational morphism $\beta : S \to \mathbb{P}^1 \times \mathbb{P}^1$ such that
$E_1, \cdots, E_4$: the exceptional divisors contracted by $\beta$,
$F, F'$: general fibers of two conic fibrations $S \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$

$$2F.\alpha \geq \sum_{i=1}^{4} E_i.\alpha, \quad 2F'.\alpha \geq \sum_{i=1}^{4} E_i.\alpha$$

Let us fix such a birational morphism.

We set

$$a(\alpha) = F.\alpha,\ a'(\alpha) = F'.\alpha,\ k_i(\alpha) = E_i.\alpha.$$

# Fibrations induced by a conic bundle

$B = \mathbb{P}^1$

$U_{\mathbf{k}}$: the Zariski open subset of $\prod_{i=1}^4 \mathrm{Hilb}^{[k_i]}(B)$ parametrizing $([T_i])$ such that the supports of $T_i$ are mutually disjoint.

Then we consider the following morphism:

$$\Phi_\alpha : \mathrm{Mor}(B, S, \alpha) \to U_{\mathbf{k}}, [s : B \to S] \mapsto (s^* E_i)_{i=1}^4$$

Using some deformation theory,
one can prove that this is dominant, and is an open subset of a $\mathbb{P}^{2a+1-\sum_i k_i} \times \mathbb{P}^{2a'+1-\sum_i k_i}$-bundle over $U_{\mathbf{k}}$.

# Rationality

**Theorem (T)**

*Assume* $(-K_S)^2 = 4$ *and* $S$ *is split.*
*Let* $\alpha$ *be a nef class on* $S$*.*

*Then* $\mathrm{Mor}(\mathbb{P}^1, S, \alpha)$ *is rational.*

**Remark**

*Fanelli–Gruson–Perrin showed that*

$$M_{\mathrm{bir}}(S, \alpha)$$

*is unirational for some* $\alpha$*, and we proved it for all of them.*

# Upper bounds

**Theorem (Das–Lehmann–T–Tosteson)**

*Over $\mathbb{F}_q$, assume that $S$ is split. Then*

$$\limsup_{d\to\infty} \frac{N(\mathbb{P}^1, S, -K_S, d)}{q^d d^5} \leq \frac{\alpha(-K_S)q^2}{(1-q^{-1})^7}.$$

**Remark**

*Previously Glas showed that*

$$N(\mathbb{P}^1, S, -K_S, d) = o(q^{d(1+\epsilon)}),$$

*for any $\epsilon > 0$.*

# The method of the bar complexes

## III. The method of the bar complexes

## Set up

$\alpha$: a nef class on $S$
Recall that we fix a birational morphism

$$\beta : S \to \mathbb{P}^1 \times \mathbb{P}^1,$$

with divisors $F, F', E_1, E_2, E_3, E_4$.

$(p_i, p'_i) \in \mathbb{P}^1 \times \mathbb{P}^1$: the blow up points of $\beta$
Then $M_\alpha = \mathrm{Mor}(B, S, \alpha)$ can be identified with the space of morphisms

$$B \to \mathbb{P}^1 \times \mathbb{P}^1$$

of degree $(a, a') = (F.\alpha, F'.\alpha)$ such that the multiplicity at $(p_i, p'_i)$ is equal to $k_i = E_i.\alpha$

## Set up

$\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}(V_1) \times \mathbb{P}(V_2)$ with $V_i$ is a 2-dimensional space over $k$
$\ell_{i,j} \subset V_j$: the 1-dimensional subspaces corresponding to $p_i, p_i'$.

We consider the space $\widetilde{M}_\alpha$ of sections

$$(s, t) \in \Gamma(B, V_1 \otimes \mathcal{O}(a)) \oplus \Gamma(B, V_2 \otimes \mathcal{O}(a')),$$

such that

- $s, t$ are no-where vanishing
- the length of $(s,t)^{-1}(\ell_{i,1} \otimes \mathcal{O}(a) \oplus \ell_{i,2} \otimes \mathcal{O}(a'))$ is $k_i$ for $i = 1, \cdots, 4$.

Then there is a natural morphism

$$\widetilde{M}_\alpha \to M_\alpha,$$

which is realized as a $\mathbb{G}_m^2$-torsor over $M_\alpha$.

## Set up

$$\#\widetilde{M}_\alpha(k) = (q-1)^2\#M_\alpha(k)$$

$\widetilde{M}_\alpha \to M_\alpha \to U_{\mathbf{k}}$: a dominant morphism
for each $w \in U_{\mathbf{k}}$, the fiber $\widetilde{M}_{\alpha,w}$ is a Zariski open subset of

$$E_w = \Gamma(B, V_1 \otimes \mathcal{O}(a) \oplus V_2 \otimes \mathcal{O}(a'))_w,$$

which is the space of sections $(s,t)$ with the incidence conditions

$$s(w_i) \subset \ell_{i,1} \otimes \mathcal{O}(a), \text{ and } t(w_i) \subset \ell_{i,2} \otimes \mathcal{O}(a').$$

Thus, $\widetilde{M}_\alpha$ is a Zariski open subset of a vector bundle $E \to U_{\mathbf{k}}$.
Our goal is to understand # of $k$-points on the complement

$$E \setminus \widetilde{M}_\alpha.$$

This is a reason why the inclusion-exclusion principle is kicked in i.e., the bar complexes developed by Das–Tosteson over $\mathbb{C}$.

## A stratification

$Q$: the poscheme (a scheme with a poset structure) defined by

$$Q = \{V = V_1 \oplus V_2, V_1, V_2, \ell_{i,1} \oplus \ell_{i,2}, \ell_{i,1} \oplus \{0\}, \{0\} \oplus \ell_{i,2}, \{0\}\},$$

where the poset structure is given by inclusion of subspaces.
$\mathrm{Hilb}(B)$: the Hilbert scheme as a poscheme
where the poset structure is given again by inclusion.
$\mathrm{Hilb}(B)^Q$: the scheme of homomorphisms $x : Q \to \mathrm{Hilb}(B)$ s.t. $x^{-1}(B) = V$.

Then the complement $E \setminus \widetilde{M}_\alpha$ admits a stratification defined by for any $(w < x) \in (U_{\mathbf{k}} < \mathrm{Hilb}(B)^Q)(\overline{k})$, a subspace

$$Z_{w<x} = \Gamma(B, V_1 \otimes \mathcal{O}(a) \oplus V_2 \otimes \mathcal{O}(a'))_x \subset E_w,$$

and these subspaces are bundled as

$$Z \subset (U_{\mathbf{k}} < \mathrm{Hilb}(B)^Q) \times_{U_{\mathbf{k}}} E.$$

# the bar complex

$\exists$ a stratification $(U_{\mathbf{k}} < \mathrm{Hilb}(B)^Q) = \sqcup_T \mathcal{N}_T$
into locally closed subsets parametrized by combinatorial types $T$

$P \subset (U_{\mathbf{k}} < \mathrm{Hilb}(B)^Q)$: a certain closed subscheme such that it is downward closed and it is a union of $\mathcal{N}_T$ of finitely many combinatorial types $T$.

The bar complex $B(P, Z)$ is a simplicial scheme which is a contravariant functor from the category $\Delta$ of non-empty finite sets of $\mathbb{N}$ with non-decreasing maps to the category of schemes:

$$[n] = \{0, \ldots, n\} \mapsto$$
$$B(P, Z)([n]) = \{w < x_0 \leq \ldots \leq x_n, z \in Z_{w<x_n} \,|\, (w < x_i) \in P\}.$$

The following theorem has been proved over $\mathbb{C}$ by Das–Tosteson, and we establish this over an arbitrary perfect field:

# the bar complex

**Theorem (Das–Lehmann–T–Tosteson, '25)**

*Let $l \in \mathbb{N}$. Assume that*

1. $P \to U_{\mathbf{k}}$ *is proper;*
2. *for every pair* $(w < x) \in P(\overline{k})$ *with* $x$ *being saturated and every saturated type* $y$ *such that* $x \prec y$*, the fiber* $Z_{w<y}$ *has expected dimension, and;*
3. $P$ *contains every* $\mathcal{N}_T$ *such that* $\kappa(T) \leq l$*,*

*where* $\kappa(T)$ *is a certain combinatorial function.*
*Then for all* $i > 2\mathrm{dim}(E) - l - 2$*, we have*

$$H^i_{ét,c}(\mathrm{im}((Z)_{\overline{k}} \to (E)_{\overline{k}}), \mathbb{Q}_\ell) \cong H^i_{ét,c}(B(P,Z)_{\overline{k}}, \mathbb{Q}_\ell).$$

The assumptions of the above theorem are verified by the description of the moduli space $M_\alpha$ mentioned before with $l = \lfloor \frac{1}{8} \min\{2a + 1 - \sum_i k_i, 2a' + 1 - \sum_i k_i\} - \frac{1}{2} \rfloor$.

# Homological sieve

By the Grothendieck–Lefschetz trace formula,
our goal is to understand, up to the error terms,

$$\sum_{i \geq 4a+4a'-2\sum_j k_j - l + 2} (-1)^i \mathrm{Tr}(\mathrm{Frob} \curvearrowright H^i_{\text{ét},c}(B(P,Z)_{\overline{k}}, \mathbb{Q}_\ell)).$$

By a spectral sequence associated to a stratification of $B(P,Z)$ + Grothendieck–Lefschetz trace formula for simplicial schemes (homological sieve), again up to the error terms, the above quantity is equal to

$$-q^{2a+2a'+4} \sum_{(w<x) \in (U_{\mathbf{k}} \leq Q^{JB})(k)} \mu_k(w,x) q^{-\gamma(x)},$$

$Q^{JB} \subset \mathrm{Hilb}(B)^Q$: Zariski open consisting of saturated elements,
$\mu_k$: the Möbius function for the poset $Q^{JB}(k)$,
$\gamma(x)$: the expected codimension of the incidence condition by $x$.

## Homological sieve

Putting altogether, the quantity $\#\widetilde{M}_\alpha(k)$ is equal to

$$q^{2a+2a'+4}\sum_{(w\leq x)\in(U_{\mathbf{k}}\leq Q^{JB})(k)}\mu_k(w,x)q^{-\gamma(x)},$$

up to the error terms.

# The virtual height zeta functions and Peyre's constant

## IV. The virtual height zeta functions and Peyre's constant

## Expectation

We showed

$$\#\widetilde{M}_{\alpha}(k) \sim q^{2a+2a'+4} \sum_{(w\leq x)\in(U_{\mathbf{k}}\leq Q^{JB})(k)} \mu_k(w,x)q^{-\gamma(x)},$$

as $a, a', k_i \to \infty$. The expectation is

$$\begin{aligned} &\#\widetilde{M}_{\alpha}(k) \\ &\sim (q-1)^2\tau_{-K_S}(S)q^{2a+2a'-\sum_i k_i} \\ &= (1-q^{-1})^{-4}\left(\prod_{c\in|B|}(1-q^{-|c|})^6(1+6q^{-|c|}+q^{-2|c|})\right) \\ &\qquad \times q^{2a+2a'-\sum_i k_i+4}, \end{aligned}$$

as $a, a', k_i \to \infty$.

## The virtual height zeta function

To this end, we consider the following virtual height zeta function:

$$Z(\mathbf{t}) = \sum_{k_1,\cdots,k_4=0}^{\infty} q^{\sum_{i=1}^{4} k_i} \left( \sum_{(w \leq x) \in (U_{\mathbf{k}} \leq Q^{JB})(k)} \mu_k(w,x) q^{-\gamma(x)} \right) t_1^{k_1} t_2^{k_2} t_3^{k_3} t_4^{k_4}.$$

Since the Möbius function is multiplicative, this becomes the Euler product

$$Z(\mathbf{t}) = \prod_{c \in |B|} \Bigg( 1 - 6q^{-2|c|} + 8q^{-3|c|} - 3q^{-4|c|} + \sum_{i=1}^{4} \sum_{d=1}^{\infty} (qt_i)^{|c|d} \left( q^{-2d|c|} - 2q^{-(2d+1)|c|} + 2q^{-(2d+3)|c|} - q^{-(2d+4)|c|} \right) \Bigg).$$

# The virtual height zeta function

**Proposition**

$$\lim_{t_i\to 1}\prod_{i=1}^{4}(1-t_i)\cdot \mathrm{Z}(\mathbf{t}) = (1-q^{-1})^{-4}\prod_{c\in|B|}(1-q^{-|c|})^6(1+6q^{-|c|}+q^{-2|c|}).$$

Thus Abel's summation will prove

$$q^{2a+2a'+4}\sum_{(w\le x)\in(U_{\mathbf{k}}\le Q^{JB})(k)}\mu_k(w,x)q^{-\gamma(x)}\sim$$

$$(1-q^{-1})^{-4}\left(\prod_{c\in|B|}(1-q^{-|c|})^6(1+6q^{-|c|}+q^{-2|c|})\right)q^{2a+2a'-\sum_i k_i+4}$$

$$=(q-1)^2\tau_{-K_S}(S)q^{2a+2a'-\sum_i k_i},$$

as $a, a', k_i \to \infty$.

## Error terms

**Theorem (Sawin–Shusterman)**

*There exists a constant $C > 1$ such that for any $\alpha$ we have*

$$\sum_{i=0}^{\infty} \dim H^{i}_{\acute{e}t,c}(\mathrm{Mor}(B,S,\alpha),\mathbb{Q}_\ell) \le C^{-K_S.\alpha+1}.$$

This is based on Katz's results on upper bounds for betti numbers of affine varieties in terms of degrees and the number of equations.

We need to assume that $q^\epsilon > C$ to beat this error estimate.

## Homological stability

# V. Homological stability

# Cohen-Jones-Segal conjecture

We work over $\mathbb{C}$.
$\mathrm{Top}_*(\mathbb{P}^1, S, \alpha)$:
The space of continuous pointed maps $s : \mathbb{P}^1(\mathbb{C}) \to S(\mathbb{C})$

$$\mathrm{Mor}_*(\mathbb{P}^1, S, \alpha) \hookrightarrow \mathrm{Top}_*(\mathbb{P}^1, S, \alpha).$$

Cohen–Jones–Segal conjecture predicts that this should approximate the topology.

## Cohen–Jones–Segal conjecture

**Theorem (Das–Lehmann–T–Tosteson, '25)**

*$S$: a smooth del Pezzo surface of degree 4*

*Then there exists a homogeneous and piecewise linear function $\ell$ on $\mathrm{Nef}_1(S)$ which is positive on a dense open cone $U \subset \mathrm{Nef}_1(S)$ and $c > 0$ such that*

$$H_i(\mathrm{Mor}_*(\mathbb{P}^1, S, \alpha), \mathbb{Z}) \cong H_i(\mathrm{Top}_*(\mathbb{P}^1, S, \alpha), \mathbb{Z}),$$

*for $i < \ell(\alpha) - c$.*

Previously Das–Tosteson obtained such a statement for quintic del Pezzo surfaces.

# Homological stability and weak approximation (Joint with Yuri Tschinkel)

$B$: a smooth projective curve over a field $\mathbb{C}$
$F = \mathbb{C}(B)$ its function field.

$X$: a smooth Fano variety over $F$

Let

$$\pi : \mathcal{X} \to B$$

its smooth integral model, i.e.,
a flat proper morphism from a smooth projective $\mathcal{X}$ over $\mathbb{C}$,
with generic fiber $X$.

# Homological stability and weak approximation (Joint with Yuri Tschinkel)

$\alpha \in H_2(\mathcal{X}(\mathbb{C}), \mathbb{Z})$: the class of a section of $\pi$
$\beta \in H_2(\mathcal{X}(\mathbb{C}), \mathbb{Z})$: the class of a very free rational curve in a smooth fiber of $\pi$.

$\Sigma = \{\hat{\sigma}_j\}_{j \in J}$: a finite set of jets on $\mathcal{X}$ in distinct fibers $\mathcal{X}_{b_j}$, where each $\hat{\sigma}_j$ is an admissible $N_j$-th jet.

Let

$$\mathrm{Sec}(\mathcal{X}/B, \alpha, \Sigma) :=$$

$$\{\sigma : B \to ;X \mid [\sigma(B)] = \alpha, \quad \sigma(b_j) \equiv \hat{\sigma}_j \pmod{\mathfrak{m}_{B,b_j}^{N_j+1}}, \quad \forall j \in J\}$$

the space of sections of class $\alpha$ matching all jets in $\Sigma$

# Homological stability and weak approximation (Joint with Yuri Tschinkel)

**Condition (HS)** (Homological stability, for the triple $(\alpha, \beta, \Sigma)$): For $\alpha, \beta$, and $\Sigma$ as above, there exists a linear function $\ell$, with positive leading term, such that, for all $i \leq \ell(m)$, one has

$$H_i(\mathrm{Sec}(\mathcal{X}/B, \alpha + m\beta, \Sigma), \mathbb{Z}) \simeq H_i(\mathrm{Sec}(\mathcal{X}/B, \alpha + (m+1)\beta, \Sigma), \mathbb{Z}).$$

## Theorem (T–Tschinkel, '25)

*Homological stability, for any $(\alpha, \beta, \Sigma)$ with nonempty $\Sigma$, holds for:*

- $\mathbb{P}(\mathcal{E})$, *projectivization of a vector bundle $\mathcal{E}$ of rank $\geq 2$ over $B$,*
- *smooth conic bundles over $B$,*
- *smooth nonsplit quadric surface bundles over $B$, with at most $\mathrm{A}_1$-singular fibers.*

Thank you!!