Characterization of products of projective spaces via the nef complexity

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§ 1 Intro

Setting (Unless otherwise stated)

X : sm. Fano var. of dim n / $\mathbb{C}$

Def $i_X := \max\{m \in \mathbb{Z}_{>0} \mid \frac{1}{m}(-K_X) \in \operatorname{Pic} X\}$

Fano index

Kobayashi-Ochiai Thm (i) $i_X \leq n+1$

(ii) $i_X = n+1 \iff X \simeq \mathbb{P}^n$

(iii) $i_X = n \iff X \simeq Q^n$

$i_X = n-1$ del Pezzo var.

$i_X = n-2$ Mukai var.

Mukai Conj $\rho_X(i_X - 1) \leq n$ $\leftarrow$ $n + \rho_X - \underbrace{i_X \rho_X}_{} \geq 0$

Moreover "$=$" $\iff$ $X \simeq (\mathbb{P}^{i_X - 1})^{\rho_X}$

Replace this part by the "total index τ_X".

Def For $R = \mathbb{Z}$ or $\mathbb{Q}$,

$$\tau_X(R) := \max\left\{ \sum_{i=1}^{k} a_i \,\middle|\, -K_X \equiv \sum_{i=1}^{k} a_i L_i,\ a_i \in R_{>0},\ \underset{\substack{\not\equiv \\ 0}}{L_i} : \text{nef Car} \right\}$$

$\tau_X := \tau_X(\mathbb{Q})$: the total index of X

$c_X := n + \rho_X - \tau_X$: the nef complexity of X

Mukai-type Conj. (Gongyo '23)

- $c_X \geq 0$
- "=" $\iff X \simeq \prod \mathbb{P}^{n_i}$

Known Results (i) $\dim X = 2$ (Gongyo-Moraga '23)

(ii) The Ambro-Kawamata effective nonvanishing Conj.

$\Rightarrow$ The Mukai-type Conj. (Gongyo '23)

(iii) X : spherical (Gagliardi-Hofscheier-Pearson '25)

Main Thm The Mukai-type Conj. holds.

Thm $(X, \Delta + \sum m_i M_i)$: generalized klt pair with

$$K_X + \Delta + \sum m_i M_i \equiv 0$$

where M_i is nef Cartier & $M_i \not\equiv 0$ & $\Delta + \sum m_i M_i$: big

$\Rightarrow$ · $\sum m_i \leq \dim X + \rho_X$

· "$=$" $\iff$ $X \simeq \prod \mathbb{P}^{n_i}$

Cor The Mukai-type Conj holds for klt Fano var.s.

Thm (a positive answer to a question by J. Starr)

X : sm. Fano var s.t. $^\forall$elem. contr. is of fiber type —— (*)

$\Longrightarrow$ $NE(X)$ & $Nef(X)$ are simplicial.

Thm The Mukai Conj holds for (*).

Other results we have proved

- Computation of I_X for sm. del Pezzo surfs.
- Classification of Fano 3-folds with $c_X = 1$.
- $\prod \mathbb{P}^{n_i} \twoheadrightarrow \underset{\sim \text{sm. proj. var.}}{Y} \Longrightarrow Y \simeq \prod \mathbb{P}^{m_i}$
- Results on "polytopes"

§ 2 Examples

Setting X : sm. Fano var of dim n. / $\mathbb{C}$

e.g. $\rho_X = 1 \Longrightarrow \tau_X = i_X$

[($\because$) $\operatorname{Pic} X = \mathbb{Z}[H]$ for some ample H.

$-K_X = i_X H \rightsquigarrow \tau_X = i_X$.]

e.g. $X = X_1 \times X_2 \Longrightarrow \tau_X = \tau_{X_1} + \tau_{X_2}$

e.g. Assume $\operatorname{Nef}(X) \cap \operatorname{Pic}(X) = \sum_{i=1}^{\rho_X} \mathbb{Z}_{\geq 0} [L_i]$.

If $-K_X = \sum a_i L_i$, then $\tau_X = \sum a_i$

e.g. For a linear subsp. $\mathbb{P}^k \subset \mathbb{P}^n$,

$$\begin{tikzcd} X = \mathrm{Bl}_{\mathbb{P}^k}(\mathbb{P}^n) \arrow[r, "\varphi"'] \arrow[d, "\pi"'] & \mathbb{P}^n \arrow[dl, dotted, bend left] \\ \mathbb{P}^{n-k-1} & \end{tikzcd}$$

$H := \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$

E : exc. div.

$\pi^* \mathcal{O}_{\mathbb{P}^{n-k-1}}(1) = H - E$

$\mathrm{Nef}(X) = \mathbb{R}_{\geq 0}[H] + \mathbb{R}_{\geq 0}[H-E]$

$$\mathrm{Nef}(X) \cap \mathrm{Pic}(X) = \mathbb{Z}_{\geq 0}[H] + \mathbb{Z}_{\geq 0}[H-E]$$

$$-K_X = (n+1)H - (n-k-1)E = (k+2)H + (n-k-1)(H-E)$$

$$\rightsquigarrow \quad \tau_X = (k+2) + (n-k-1) = n+1$$

$$c_X = n + \rho_X - \tau_X = n+2-(n+1) = 1$$

sm. del Pezzo surf.

Notation $\pi : X = \mathrm{Bl}_r(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$: bl up of r points $P_1, \ldots, P_r$ in general position

$E_i := \pi^{-1}(P_i)$, $H := \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$ $(1 \le r \le 8)$

Thm For a sm. dP surf X,

$$\tau_X(\mathbb{Q}) = \begin{cases} 3 & X = \mathbb{P}^2 \\ 4 & X = \mathbb{P}^1 \times \mathbb{P}^1 \\ 3 & X = \mathrm{Bl}_r(\mathbb{P}^2) \quad r = 1,2,3 \\ \frac{9-r}{2} & X = \mathrm{Bl}_r(\mathbb{P}^2) \quad r = 4,5,6,7 \\ 1 & X = \mathrm{Bl}_8(\mathbb{P}^2) \end{cases}$$

$$\tau_X(\mathbb{Z}) = \begin{cases} 3 & X = \mathbb{P}^2 \\ 4 & X = \mathbb{P}^1 \times \mathbb{P}^1 \\ 3 & X = \mathrm{Bl}_r(\mathbb{P}^2) \quad r = 1,2,3 \\ 2 & X = \mathrm{Bl}_r(\mathbb{P}^2) \quad r = 4,5 \\ 1 & X = \mathrm{Bl}_r(\mathbb{P}^2) \quad r = 6,7,8 \end{cases}$$

Key Prop Set $m(X) := \min\{-K_X \cdot L \mid L \underset{\not\equiv 0}{:} \text{nef Car.}\}$

$$\Phi_{\text{base}}(L) := \frac{1}{m(X)}(-K_X \cdot L)$$ base certificate

Then for $^\forall$nef Cartier $L \not\equiv 0$, $\Phi_{\text{base}}(L) \geq 1$.

Consequently, $\tau_X(\mathbb{Q}) \leq \dfrac{(-K_X)^2}{m(X)}$, $\tau_X(\mathbb{Z}) \leq \left\lfloor \dfrac{(-K_X)^2}{m(X)} \right\rfloor$

Pf By definition of $m(X)$, $(-K_X \cdot L) \geq m(X)$ for $^\forall$nef Car. $L \not\equiv 0$

$\rightsquigarrow$ $\Phi_{\text{base}}(L) \geq 1$

nef Cartier $\not\equiv 0$

If $-K_X = \sum a_i L_i$, then

$$\sum a_i \leq \sum a_i \Phi_{\text{base}}(L_i) = \Phi_{\text{base}}\left(\sum a_i L_i\right) = \Phi_{\text{base}}(-K_X) = \frac{(-K_X)^2}{m(X)}$$

//

How to compute $\mathcal{T}_X$

Assume $X = \mathrm{Bl}_r(\mathbb{P}^2)$ $(r \geq 3)$

Compute

$$m(X) = \begin{cases} 2 & (3 \leq r \leq 7) \\ 1 & (r = 8) \end{cases}$$

$$(-K_X)^2 = 9 - r$$

Key Prop

$$\mathcal{T}_X(\mathbb{Q}) \leq \frac{(-K_X)^2}{m(X)} = \begin{cases} \frac{9-r}{2} & (3 \leq r \leq 7) \\ 1 & (r = 8) \end{cases}$$

For $^\forall$4-subset $I \subset \{1, \dots, r\}$,

$f_{[I]}$: the strict transf. of a conic through 4 pts $\{P_i\}_{i \in I}$

$\approx 2H - \sum_{i \in I} E_i \in \mathrm{Pic}(X)$

$\underline{r=3}$ $\quad -K_X = (H-E_1)+(H-E_2)+(H-E_3)$ $\quad \underline{3}$

$\underline{r=4}$ $\quad -K_X = \frac{1}{2}\underset{\substack{\| \\ 2H-E_1-E_2-E_3-E_4}}{f_{[1234]}} + \sum_{i=1}^{4}\frac{1}{2}(H-E_i)$ $\leftarrow$ $\mathbb{Q}$-decomp.

$\frac{1}{2}+4\times\frac{1}{2}=\underline{\frac{5}{2}}$

$= f_{[1234]} + H$ $\leftarrow$ $\mathbb{Z}$-decomp. $\quad \underline{2}$

$\underline{r=5}$ $\quad -K_X = f_{[1234]} + (H-E_5)$ $\quad \underline{2}$

$\underline{r=6}$ $\quad -K_X = \frac{1}{2}f_{[1234]} + \frac{1}{2}f_{[3456]} + \frac{1}{2}f_{[1256]}$ $\leftarrow$ $\mathbb{Q}$-dec.

$\underline{\frac{3}{2}}$

$= -K_X$ $\leftarrow$ $\mathbb{Z}$-decomp. $\quad \underline{1}$

$\underline{r=7}$ $\quad -K_X$: nef $\quad \underline{1}$

§3 Idea of the proof

Numerical Fujita-type Thm X : sm. proj. var. of dim n.

$L_1, \cdots, L_k$: nef big Cartier div.s on X, $a_i \in \mathbb{R}_{>0}$.

$$\sum_{i=1}^{k} a_i \geq n+1 \implies K_X + a_1 L_1 + \cdots + a_k L_k : \text{pseudo-eff.}$$

{ using this

Kobayashi-Ochiai-type characterization

If X is a klt Fano & $-K_X = \sum a_i M_i$ with M_i : nef big

$\sum a_i \geq n+1$, then $\sum a_i = n+1$ & $X \simeq \mathbb{P}^n$.

Thm $(X, \Delta + \sum m_i M_i)$: generalized klt pair with

$$K_X + \Delta + \sum m_i M_i \equiv 0$$

where M_i is nef Cartier & $M_i \not\equiv 0$ & $\Delta + \sum m_i M_i$: big

$\Rightarrow \sum m_i \le \dim X + \rho_X$

"=" $\Longleftrightarrow X \simeq \prod \mathbb{P}^{n_i}$

The proof only applies to generalized klt Calabi-Yau pairs, but to explain the idea of the proof in a simple setting, let X be a sm. Fano var. & $-K_X = \sum a_i L_i$ $(a_i \in \mathbb{Q}_{>0})$

L_i : nef Cartier on X s.t. $\tau_X = \sum a_i$

$\not\equiv$
0

Goal If $c_X \le 0$, $X \simeq \prod \mathbb{P}^{n_i}$

$\hookrightarrow$ $n + \rho_X \le \tau_X = \sum a_i$

$$\begin{array}{c} n+\rho_X \\ \vee\!| \\ n+1 \end{array}$$

If L_i is big for $^\forall i$, KO-type characterization implies $X \simeq \mathbb{P}^n$

So assume $^\exists i_0$ s.t. L_{i_0} is NOT big.

$i_0 = 1$ WLOG

X : Fano $\rightsquigarrow$ L_1 : semiample

$f := \mathrm{cont}_{|mL_1|} : X \longrightarrow T$: contr. of fiber type.

By induction, we obtain inequalities / T & F : gen. fiber of f,

Thank you
very much!!